28

Geometric Probability

HERBERT SOLOMON
Stanford University
Stanford, California

CBMS-NSF
REGIONAL CONFERENCE SERIES
IN APPLIED MATHEMATICS

SPONSORED BY
CONFERENCE BOARD OF
THE MATHEMATICAL SCIENCES

SUPPORTED BY
NATIONAL SCIENCE
FOUNDATION

CBMS-NSF REGIONAL CONFERENCE SERIES IN APPLIED MATHEMATICS

A series of lectures on topics of current research interest in applied mathematics under the direction of the Conference Board of the Mathematical Sciences, supported by the National Science Foundation and published by SIAM.

GARRETT BIRKHOFF, *The Numerical Solution of Elliptic Equations*
D. V. LINDLEY, *Bayesian Statistics, A Review*
R. S. VARGA, *Functional Analysis and Approximation Theory in Numerical Analysis*
R. R. BAHADUR, *Some Limit Theorems in Statistics*
PATRICK BILLINGSLEY, *Weak Convergence of Measures: Applications in Probability*
J. L. LIONS, *Some Aspects of the Optimal Control of Distributed Parameter Systems*
ROGER PENROSE, *Techniques of Differential Topology in Relativity*
HERMAN CHERNOFF, *Sequential Analysis and Optimal Design*
J. DURBIN, *Distribution Theory for Tests Based on the Sample Distribution Function*
SOL I. RUBINOW, *Mathematical Problems in the Biological Sciences*
P. D. LAX, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*
I. J. SCHOENBERG, *Cardinal Spline Interpolation*
IVAN SINGER, *The Theory of Best Approximation and Functional Analysis*
WERNER C. RHEINBOLDT, *Methods of Solving Systems of Nonlinear Equations*
HANS F. WEINBERGER, *Variational Methods for Eigenvalue Approximation*
R. TYRRELL ROCKAFELLAR, *Conjugate Duality and Optimization*
SIR JAMES LIGHTHILL, *Mathematical Biofluiddynamics*
GERARD SALTON, *Theory of Indexing*
CATHLEEN S. MORAWETZ, *Notes on Time Decay and Scattering for Some Hyperbolic Problems*
F. HOPPENSTEADT, *Mathematical Theories of Populations: Demographics, Genetics and Epidemics*
RICHARD ASKEY, *Orthogonal Polynomials and Special Functions*
L. E. PAYNE, *Improperly Posed Problems in Partial Differential Equations*
S. ROSEN, *Lectures on the Measurement and Evaluation of the Performance of Computing Systems*
HERBERT B. KELLER, *Numerical Solution of Two Point Boundary Value Problems*
J. P. LASALLE, *The Stability of Dynamical Systems* - Z. ARTSTEIN, *Appendix A: Limiting Equations and Stability of Nonautonomous Ordinary Differential Equations*
D. GOTTLIEB AND S. A. ORSZAG, *Numerical Analysis of Spectral Methods: Theory and Applications*
PETER J. HUBER, *Robust Statistical Procedures*
HERBERT SOLOMON, *Geometric Probability*
FRED S. ROBERTS, *Graph Theory and Its Applications to Problems of Society*
JURIS HARTMANIS, *Feasible Computations and Provable Complexity Properties*
ZOHAR MANNA, *Lectures on the Logic of Computer Programming*
ELLIS L. JOHNSON, *Integer Programming: Facets, Subadditivity, and Duality for Group and Semi-Group Problems*
SHMUEL WINOGRAD, *Arithmetic Complexity of Computations*
J. F. C. KINGMAN, *Mathematics of Genetic Diversity*
MORTON E. GURTIN, *Topics in Finite Elasticity*
THOMAS G. KURTZ, *Approximation of Population Processes*

(continued on inside back cover)

Herbert Solomon
Stanford University
Stanford, California

Geometric Probability

siam.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

PHILADELPHIA

This page intentionally left blank

Geometric Probability

10 9 8 7 6 5 4 3

ISBN 0-89871-025-1

Contents

Preface vii

Chapter 1
BUFFON NEEDLE PROBLEM, EXTENSIONS, AND ESTIMATION OF π
The Buffon needle problem 2
Laplace extension of Buffon problem 3
Statistical estimation procedures 6
Buffon's problem with a long needle 10
General extension of Buffon problem 13
Buffon problem in three dimensions and an application 18

Chapter 2
DENSITY AND MEASURE FOR RANDOM GEOMETRIC ELEMENTS
Straight lines in the plane 28
Pairs of points in the plane 32

Chapter 3
RANDOM LINES IN THE PLANE AND APPLICATIONS
Poisson field for random lines in the plane 44
Higher order moments for random polygonal areas 48
Anisotropic lines in the plane: An application 55
Additional results for anisotropic random lines 61

Chapter 4
COVERING A CIRCLE CIRCUMFERENCE AND A SPHERE SURFACE
Random arcs on the circumference of a circle 76
Moments of coverage 82
Distribution of coverage 84
Random caps on a sphere 85
Application of method of moments 89

Chapter 5
CROFTON'S THEOREM AND SYLVESTER'S PROBLEM IN TWO AND THREE DIMENSIONS
Crofton's theorem on mean values 97
Sylvester's four point problem 101
Sylvester's problem in three dimensions 117

Chapter 6
RANDOM CHORDS IN THE CIRCLE AND THE SPHERE
Random chords in the circle 133
More results on random chords 151
Random chords of a sphere 161

References 173

Preface

I was invited by the Mathematics Department of the University of Nevada, Las Vegas to give ten lectures on geometrical probability during the week June 9–13, 1975. Those invited to attend the lectures were given an extensive set of notes to go along with the presentations. This monograph is a version of the content of those lectures and the lecture notes.

Geometrical probability has enjoyed a resurgence of interest in recent years. The lectures and subsequently this monograph in no way cover all topics that could fall legitimately under such a heading. The material here is on topics I have presented in several classes in geometrical probability at Stanford University. Several students who attended these classes have written papers and doctoral dissertations on the subject and in this way have helped me prepare this exposition. In particular I would like to single out David Berengut, Peter Cooke, Stuart Dufour and Andrew F. Siegel whose efforts have made possible several sections of the monograph. My thanks are offered to my colleague, Persi Diaconis, for editing and reviewing the early portions of the manuscript.

I am very grateful to Dr. Aaron Goldman of the University of Nevada for initiating and directing the Conference. Participants at the Conference exhibited a lively interest in the subject and I am grateful to them for this stimulation. My thanks go also to the Office of Naval Research and the Army Research Office for their support over the years that led to the preparation of the notes that were made available to the participants at the Conference.

This page intentionally left blank

CHAPTER 1

Buffon Needle Problem, Extensions, and Estimation of π

The Buffon needle problem which many of us encountered in our college or even high school days has now been with us for two hundred years. One major aspect of its appeal is that its solution has been tied to the value of π which can then be estimated by physical simulation of the model as was done by a number of investigators in the late 19th and early 20th centuries—and by computer simulation today. Shortly we go into some detail on enlarging the experimental design of the model. Then we discuss a number of ways in which modern statistical procedures can yield estimates of π from these experimental designs with much better precision than the original Buffon procedure employed by many for the estimation of π. In this sense we will be featuring the statistician in geometrical probability and this will continue, where possible, for other topics we discuss in this exposition.

It is interesting that in the original development, Buffon (1777) extols geometry as a companion tool to the calculus in establishing a science of probability and suggests that chance is amenable to the methods of geometry as well as those of the calculus. Buffon indicates that the human mind, because of prior mathematics, preferred numbers to measures of area but that the invention of games revolving around area and ratios of areas could rectify this. To highlight this point he investigated a game already in practice in the 18th century known as "clean tile".

In a room tiled or paved with equal tiles, of any shape, a coin is thrown upwards; one of the players bets that after its fall the coin will rest cleanly, i.e., on one tile only; the second bets that the coin will rest on two tiles, i.e., that it will cover one of the cracks which separate them; a third player bets the coin will rest over 3, 4, or 6 cracks: it is required to find the chances for each of these players.

Buffon investigates this game for square tiles, tiles shaped as equilateral triangles, hexagonal tiles, and diamond shaped tiles; in each case he is interested in the ratios of the diameter of the coin to the equal sides of the particular shaped tile that provides a fair game for each player. In effect, the thrust of his work on geometrical probability is the development of fair games.

As a special case, Buffon in his own words states, "I assume that in a room, the floor of which is merely divided by parallel lines, a stick is thrown upwards and one of the players bets the stick will not interesect any of the parallels on the floor, whereas on the contrary the other one bets the stick will intersect some

one of these lines; it is required to find the chances of the two players. It is possible to play this game with a sewing needle or a headless pin."

He then demonstrates that for a fair game between two players, the ratio of the length of the needle, l, to the distance between the parallel lines, d, $(d>l)$ must equal $\pi/4$ for this provides the probability of an intersection equal to $\frac{1}{2}$. This can be seen easily from the Buffon needle result as we now know it, namely the probability of an intersection, $p=2l/(\pi d)$.

In the Buffon model a needle (line segment) is dropped "at random" on the grid of equidistant parallel lines in the plane. The notion that random elements are geometric objects such as line segments, lines in the plane, circles, rectangles, triangles, etc. requires that a measure be defined for such elements before probabilistic assertions can be made. The Bertrand paradox at the turn of the century suggested a situation wherein future developments could be stymied because of a lack of a natural choice of measure. In that paradox, the probability that a random chord in a circle exceeds the side of an inscribed equilateral triangle can be shown to be $\frac{1}{4}$, $\frac{1}{3}$, or $\frac{1}{2}$ for each of three different models by which the chord is drawn at random. Integral geometry becomes helpful here in establishing appropriate models.

Randomness models will play an important role throughout. As a second example, the question of whether pairs of chromosomes are randomly distributed in the nucleus of a cell during mitosis is translated into a geometrical probability problem regarding the expected number of intersections of n pairs of chords in a circle. Naturally, the expected number depends on the randomization model for obtaining chords in a circle and we develop six models yielding six solutions.

The Bertrand paradox led to proposals by Poincaré and others that probability statements for geometric situations be tied to densities that would be invariant under appropriate transformations. For our purposes, the group of rigid motions, that is, transformations that provide invariance under translation and rotation will serve our interest. For the Bertrand paradox, the appropriate density under the group of rigid motions leads to the solution that the probability is $\frac{1}{2}$. The other two solutions induce densities that are not invariant under the group of rigid motions.

One of the prime developers of integral geometry and its consequences for geometrical probability is L. A. Santaló. We will call on his developments and results for questions of invariance of measure and density where appropriate. Santaló has a prolific and prodigious output that is referenced in his most recent book (1976). That volume is must reading for any student of the subject for its exposition of results to date and the extensive bibliography it contains.

The Buffon needle problem. A needle (line segment) of length l is dropped "at random" on a set of equidistant parallel lines in the plane that are d units apart, $l \leqq d$.

Uspensky (1937) provides a proof that the probability of an intersection is $p=2l/(\pi d)$. He develops this by considering a finite number of possible posi-

tions for the needle as equally likely outcomes and then treats the limiting case as a representation of the problem. This includes a definition of randomness for the distance x of the needle's midpoint to the nearest line and the acute angle φ formed by the needle and a perpendicular from the midpoint to the line. The solution is obtained by computing the ratio of favorable outcomes to the total set of outcomes and passing to the limit.

A simple way of obtaining the answer is to employ a density that seems intuitively satisfactory and which turns out to be the invariant density under the group of rigid motions. This approach follows. The measure of the set of total outcomes is

$$\int_0^{\pi/2}\int_0^{d/2} dx\, d\varphi = \frac{\pi d}{4}. \tag{1.1}$$

From Fig. 1.1 we evaluate the measure of the set of favorable cases (intersections) as

$$\int_0^{\pi/2}\int_0^{(l/2)\cos\varphi} dx\, d\varphi = \frac{l}{2}; \tag{1.2}$$

therefore

$$p = \frac{l/2}{\pi d/4} = \frac{2l}{\pi d}. \tag{1.3}$$

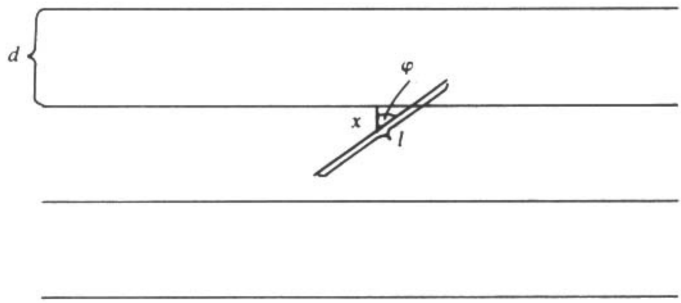


FIG. 1.1

Laplace extension of Buffon problem. Consider two sets of parallel lines over the plane where one set is orthogonal to the other. We now wish to find the probability that the needle dropped at random intersects a line of the grid. Assume the needle is shorter than the smaller sides of the congruent rectangles formed over the plane. This is the Laplace extension. First we find the probability that the needle is contained in one of the rectangles of the set.

Let a, b be the sides of the rectangle that contains the midpoint of the needle whose length is l ($l < a, l < b$). The position of the needle is determined by the coordinates x, y of its midpoint and, as before, the angle φ formed by the needle with the x-axis. Our randomness model suggests we consider x, y, φ as three

independent variables each with uniform distribution of probability over their ranges $0 \leqq x \leqq a$; $0 \leqq y \leqq b$; and $-\pi/2 \leqq \varphi \leqq \pi/2$. Thus the domain is a parallelepiped for a uniform distribution of the point x, y, φ.

The volume of the domain representing positions of the needle entirely within the rectangle is

$$V^* = \int_{-\pi/2}^{\pi/2} F(\varphi)\, d\varphi = \pi ab - 2b\pi - 2al + l^2 \tag{1.4}$$

where

$$F(\varphi) = ab - bl \cos\varphi - al|\sin\varphi| + \tfrac{1}{2}l^2|\sin 2\varphi|, \tag{1.5}$$

and the volume of the total domain $V = \pi ab$. This is developed in Uspensky (1937, p. 255).

Therefore

$$1 - p = \frac{V^*}{V} = 1 - \frac{2l(a+b) - l^2}{\pi ab} \tag{1.6}$$

and of course, the probability for the needle to intersect the perimeter of one of the rectangles is

$$p = \frac{2l(a+b) - l^2}{\pi ab}. \tag{1.7}$$

If $a = b = 1$,

$$p = \frac{4l - l^2}{\pi}. \tag{1.8}$$

This provides another approach for estimating π where $l < a$, $l < b$.

In an interesting article Schuster (1974) develops the orthogonal lines grid a bit further from the point of view of experimental design. He raises a question about the bright student who drops a needle of length L on a grid of orthogonal lines separated by distance $2L$ and repeats it until 100 observations are made of intersections with, say, lines parallel to the x-axis and these same drops are employed to count intersections with, say, lines parallel to the y-axis. How does the estimate of π from this experiment differ from that obtained by the average student who drops the needle 200 times on a grid of parallel lines separated by distance $2L$. If the x intersections and y intersections are independent, the bright student has accomplished the same purpose with half the drops.

Let A be the event-intersection with the x axis and let

$$x = \begin{cases} 1 & \text{if intersection with the } x \text{ axis,} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for B and the y axis and let

$$y = \begin{cases} 1 & \text{if intersection with the } y \text{ axis,} \\ 0 & \text{otherwise.} \end{cases}$$

We now examine $P(AB)$ and $P(A)\cdot P(B)$. From the original Buffon result we have $P(A)=P(B)=1/\pi$. We can deduce $P(AB)$ from the Laplace result as follows:

$$P(AB)=1-P(AB')-P(A'B)-P(A'B') \tag{1.9}$$

and we have shown

$$P(A'B')=1-\frac{2l(a+b)-l^2}{\pi ab}=1-\frac{2l(4l)-l^2}{4l^2\pi}=1-\frac{7}{4\pi}. \tag{1.10}$$

Thus

$$P(AB)=1-\left(1-\frac{7}{4\pi}\right)-P(AB')-P(A'B). \tag{1.11}$$

Now

$$\begin{aligned} P(A)&=\frac{1}{\pi}=P(AB)+P(AB'),\\ P(B)&=\frac{1}{\pi}=P(BA)+P(BA'),\\ P(AB')&=P(BA')=\frac{1}{\pi}-P(AB),\\ P(AB)&=\frac{7}{4\pi}-\frac{2}{\pi}+2P(AB),\\ P(AB)&=\frac{2}{\pi}-\frac{7}{4\pi}=\frac{1}{4\pi}. \end{aligned} \tag{1.12}$$

In fact, then $P(AB')=P(A'B)=3/(4\pi)$. However, we now have $P(AB)\neq P(A)P(B)$ and so there is *no* independence.

Yet we are still interested in the efficiency of

$$\hat{p}=\frac{1}{200}\sum_{i=1}^{100}(x_i+y_i). \tag{1.13}$$

Let us find Var (x_i+y_i): We have

$$\begin{aligned} \operatorname{Var}(x_i+y_i)&=\operatorname{Var} x_i+\operatorname{Var} y_i+2\operatorname{Cov} x_iy_i,\\ \operatorname{Cov} x_iy_i&=E(x_iy_i)-E(x_i)E(y_i),\\ \operatorname{Cov} x_iy_i&=P(x_i=1,\, y_i=1)-\frac{1}{\pi}\cdot\frac{1}{\pi}. \end{aligned} \tag{1.14}$$

From the previous result, $P(x_i=1,\, y_i=1)=1/(4\pi)$,

$$\operatorname{Cov} x_iy_i=\frac{1}{4\pi}-\frac{1}{\pi^2}=\frac{\pi-4}{4\pi^2}<0, \tag{1.15}$$

(1.16) $$\operatorname{Var}(x_i+y_i)=\frac{1}{\pi}\left(1-\frac{1}{\pi}\right)+\left(\frac{1}{\pi}\right)\left(1-\frac{1}{\pi}\right)+2\left\{\frac{1}{4\pi}-\frac{1}{\pi^2}\right\}=\frac{5\pi-8}{2\pi^2}$$

for $\operatorname{Var} x_i=\operatorname{Var} y_i=(1/\pi)(1-1/\pi)$. Thus

(1.17) $$\operatorname{Var}\hat{p}=\frac{1}{(200)^2}(100)\left(\frac{5\pi-8}{2\pi^2}\right)=\frac{5\pi-8}{800\pi^2}\cong 0.000976.$$

Now for the simple Buffon problem, we have

(1.18) $$\hat{\hat{p}}=\frac{1}{M}\sum_{i=1}^{M}x_i,\qquad \operatorname{Var}\hat{\hat{p}}=\frac{1}{M^2}(M)\operatorname{Var}x_i=\frac{1}{M}\frac{1}{\pi}\left(1-\frac{1}{\pi}\right),\qquad \operatorname{Var}\hat{\hat{p}}=\frac{0.217}{M}.$$

Equate $\operatorname{Var}\hat{\hat{p}}$ and $\operatorname{Var}\hat{p}$ and we get

(1.19) $$\frac{0.217}{M}=0.000976,\qquad M\cong 222.$$

Thus 100 independent observations with respect to both grids contain the same amount of information about P as 222 observations with respect to one set of grid lines.

Schuster raises another query: Is there an angle of intersection of the grid lines such that independence is achieved, i.e. an angle α such that

(1.20) $$P(AB;\alpha)=P(A;\alpha)P(B;\alpha)=\frac{1}{\pi^2}.$$

He demonstrates that

(1.21) $$P(AB;\alpha)=f(\alpha)=\{(\pi/2-\alpha)\cos\alpha+\sin\alpha\}/(4\pi)$$

and that $P(AB;\alpha)=1/\pi^2$ when $\alpha\cong 0.76605$ radians (43.89°). Let us find when $P(AB)$ is minimized. Differentiating we find that $\alpha=\pi/2$ minimizes $f(\alpha)$. Thus $1/(4\pi)$ is the minimum value for $P(AB)$ and it occurs when the grid lines are orthogonal.

Statistical estimation procedures for π in the Buffon model. Perlman and Wichura (1975) investigate a number of statistical estimation procedures for π for the single grid and the double grid and place Schuster's results and some earlier results by Mantel (1953) within the framework of their development. From Buffon we have

(1.22) $$p=\frac{2l}{\pi d}=2r\theta$$

where $\theta=1/\pi$ and $r=l/d$. It is more appropriate to estimate θ for we avoid some pitfalls in treating asymptotic variances of our estimates in this way, yet estimating $1/\pi$ or π gives us the same information. The restriction $0\leqq p\leqq 1$ imposes $0\leqq\theta\leqq 1/(2r)$. If n independent throws of the needle, l, $0\leqq l\leqq d$ result

in N crossings in the single grid then N is binomially distributed with parameters n, p and

$$(1.23) \qquad \hat{\theta}_1 = \frac{N}{2rn}$$

is an estimate for θ, and $E(\hat{\theta}_1) = \theta$. Since N is a complete sufficient statistic for θ, the Rao–Blackwell–Lehmann–Scheffé theorems imply $\hat{\theta}_1$ is a uniformly minimum variance unbiased estimator (UMVUE) of θ. Furthermore $\hat{\theta}_1$ is a maximum likelihood estimator (MLE) of θ and has 100% asymptotic efficiency in this experiment. Now

$$(1.24) \qquad \operatorname{Var} \hat{\theta}_1 = \frac{1}{4r^2} \frac{p(1-p)}{n} = \frac{\theta^2}{n}\left(\frac{1}{p} - 1\right).$$

We can minimize Var $(\hat{\theta}_1)$ by taking p as close to 1 as possible, regardless of θ, i.e. choose needle length $l = d$ for the best precision in estimating θ. In this case, $p = 2\theta$,

$$(1.25) \qquad n \operatorname{Var}(\hat{\theta}_1) = \theta^2\left(\frac{1}{2\theta} - 1\right).$$

An application of the δ-method shows that Buffon's estimator

$$(1.26) \qquad \hat{\pi}_1 = 1/\hat{\theta}_1$$

is an asymptotically unbiased 100% efficient estimator with asymptotic variance

$$(1.27) \qquad \operatorname{AVar}(\hat{\pi}_1) = \pi^4 \operatorname{Var}(\hat{\theta}_1) = \frac{\pi^2}{n}\left(\frac{\pi}{2} - 1\right) = \frac{5.63}{n}, \qquad \text{(employ } \pi = 3.1416\text{)}.$$

This result was derived previously by Mantel (1953).

For the double grid in Laplace's experiment ($l < d$) suppose we have N_A crossings of the A lines, N_B crossings of the B lines and therefore $\hat{\theta}_A = N_A/(2rn)$, $\hat{\theta}_B = N_B/(2rn)$ have the same distributions as θ_1. Schuster proposed the estimator

$$(1.28) \qquad \hat{\theta}_2 = \frac{\hat{\theta}_A + \hat{\theta}_B}{2} = \frac{N_A + N_B}{4rn}.$$

If N_A and N_B are independent then the efficiency of θ_2 is twice that of $\hat{\theta}_1$; however the efficiency is more than doubled because N_A and N_B are negatively correlated (antithetic variables). Laplace long ago obtained the crossing probabilities

$$p_A, \quad p_B, \quad p_{AB}, \quad p_{A\bar{B}}, \quad p_{\bar{A}B}, \quad p_{\bar{A}\bar{B}}$$

and these have already been discussed. We have

$$(1.29) \qquad p_{\bar{A}\bar{B}} = 1 - 4r\theta + r^2\theta, \quad \text{where } l = d = 1$$

and we know $p_A = p_B = 2r\theta$, so

(1.30) $$p_{AB} = p_A + p_B + p_{\bar{A}\bar{B}} - 1 = r^2\theta$$

and

$$1 = p_A + p_B + p_{\bar{A}\bar{B}} - p_{AB} = p_{AB} + p_{A\bar{B}} + p_{AB} + p_{\bar{A}B} + p_{\bar{A}\bar{B}} - p_{AB}.$$

Introduce the indicator

$$I_i(A) = \begin{cases} 1 & \text{if an } A\text{-line is crossed on } i\text{th throw,} \\ 0 & \text{if not} \end{cases}$$

and similarly define $I_i(B)$. Then $N_A = \sum_{i=1}^N I_i(A)$, $N_B = \sum_{i=1}^N I_i(B)$,

(1.31) $$\hat{\theta}_2 = \frac{1}{4rn} \sum_{i=1}^N [I_i(A) + I_i(B)].$$

The n pairs $[I_i(A), I_i(B)]$ are independent but $I_i(A)$ and $I_i(B)$ are dependent and so

(1.32) $$\begin{aligned} \text{Var}(\hat{\theta}_2) &= \frac{1}{16^2 rn}[\text{Var } I_i(A) + \text{Var } I_i(B) + 2 \text{ Cov}(I_i(A), I_i(B))] \\ &= \frac{\theta}{n}\left[\frac{1}{4r} + \frac{1}{8} - \theta\right]. \end{aligned}$$

Once again $\hat{\theta}_2$ has the greatest efficiency when $r = 1$ $(l = d)$. When $r = 1$, $p_{\bar{A}\bar{B}} = 1 - 3\theta$ or $0 \leqq \theta \leqq \frac{1}{3}$ or $\pi \geqq 3$. The latter inequality for π arises without any experimentation. A little later on we get sharper bounds on π without any experimentation. When $r = 1$, we have

(1.33) $$n \text{ Var}(\hat{\theta}_2^2) = \theta^2\left[\frac{3}{8\theta} - 1\right]$$

and so

(1.34) $$\text{AVar}(\hat{\pi}_2) = \frac{\pi^2}{n}\left(\frac{3\pi}{8} - 1\right) = \frac{1.76}{n}.$$

By doubling the grid we obtain an estimator $\hat{\pi}_2$, that is, $5.63/1.76 = 3.20$ times as efficient as $\hat{\pi}_1$. One throw of the needle $l = d$ on the double grid contains *at least* 3.20 times the statistical information about values of π as one throw onto the single grid.

Perlman and Wichura now ask whether one can do better. The answer is *yes* for we shall see that a complete and sufficient statistic exists for this experiment but $\hat{\theta}_2$ is *not* a function of this statistic.

Full information obtained from n throws of the needle onto the double grid can be summarized by the vector statistic

(1.35) $$\tilde{N} = (N_{AB}, N_{A\bar{B}}, N_{\bar{A}B}, N_{\bar{A}\bar{B}}).$$

Clearly $\tilde{N}$ has the multinomial distribution with probabilities

$$p_{AB}, \quad p_{A\bar{B}}, \quad p_{\bar{A}B}, \quad p_{\bar{A}\bar{B}},$$
$$p_{\bar{A}\bar{B}} = 1 - 4r\theta + r^2\theta, \qquad p_{AB} = r^2\theta.$$

Now

$$p_{\bar{A}B} = p_{A\bar{B}} = p_A - p_{AB}.$$

The probability distribution of $\tilde{N}$ is given by ($\tilde{N}$ and $\tilde{n}$ are vector-valued)

$$\begin{aligned} p_\theta(\tilde{N} = \tilde{n}) &= c(\tilde{n})(p_{AB})^{n_{AB}}(p_{\bar{A}B})^{n_{\bar{A}B}}(p_{A\bar{B}})^{n_{B\bar{A}}}(p_{\bar{A}\bar{B}})^{n_{\bar{A}\bar{B}}} \\ &= c(\tilde{n})h(\tilde{n})\theta^{(n_{AB}+n_{A\bar{B}}+n_{\bar{A}B})}(1-m\theta)^{n_{\bar{A}\bar{B}}}, \end{aligned} \tag{1.36}$$

$$c(\tilde{n}) = \frac{n!}{(n_{AB})!(n_{\bar{A}B})!(n_{A\bar{B}})!(n_{\bar{A}\bar{B}})!}, \tag{1.37}$$

$$h(\tilde{n}) = r^{2n_{AB}}[r(2-r)]^{n_{A\bar{B}}+n_{\bar{A}B}}, \tag{1.38}$$
$$m = 4r - r^2.$$

Since

$$n_{\bar{A}\bar{B}} = n - (n_{AB} + n_{\bar{A}B} + n_{A\bar{B}}) \tag{1.39}$$

the factorization implies $N_{AB} + N_{A\bar{B}} + N_{\bar{A}B}$ is a sufficient statistic for θ.

If we define N_j to be the number of times in n throws that the needle crosses exactly j lines ($j = 0, 1, 2$), with $N_0 = N_{\bar{A}\bar{B}}$, $N_1 = N_{A\bar{B}} + N_{\bar{A}B}$, $N_2 = N_{AB}$ and $\sum N_j = n$, then the sufficient statistic can be expressed as $N_1 + N_2$, the number of times in n throws that the needle crosses at least one line.

Now

$$N_1 + N_2 \sim B(n, p^*) \quad \text{where } p^* = m\theta = (4r - r^2)\theta \tag{1.40}$$

(since $p_{\bar{A}\bar{B}} = 1 - [(4r - r^2)\theta]$) so $N_1 + N_2$ is a complete as well as a sufficient statistic for θ. By the Rao–Blackwell–Lehmann–Scheffe theorems

$$\hat{\theta}_3 = \frac{N_1 + N_2}{mn} \quad \text{is UMVUE} \tag{1.41}$$

and being the MLE of θ it has 100% asymptotic efficiency in the double grid experiment.

Its variance is $\text{Var}(\hat{\theta}_3) = (\theta/n)(1/m - \theta)$ which is minimized by $l = d$ (n is maximized by $l = d$).

In this case $m = 3$, $p^* = 3\theta$, $n \text{ Var } \hat{\theta}_3 = \theta^2[1/(3\theta) - 1]$ and

$$\text{AVar}(\hat{\pi}_3) = \frac{\pi^2}{n}\left(\frac{\pi}{3} - 1\right) = \frac{0.466}{n}. \tag{1.42}$$

Thus, the fully efficient estimator $\hat{\pi}_3$ is $1.76/.466 = 3.77$ times as efficient as Schuster's estimator $\hat{\pi}_2$, reflecting the fact that $\hat{\theta}_2$ is based on

$$N_A + N_B = N_{AB} + N_{\bar{A}B} + N_{A\bar{B}} + N_{AB} = N_1 + 2N_2 \tag{1.43}$$

which is not a function of the sufficient statistic $N_1 + N_2$.

We see that antithetic variates are fine for minimizing variance but sufficient statistics when they can be found are better. In the development here we learn that a throw of the needle ($l = d$) on the double grid contains not $5.63/1.76 = 3.20$ times the statistical information as in the Schuster situation but actually $5.63/.466 = 12.08$ times the statistical information about the value of π when the sufficient statistic is employed.

Buffon's problem with a long needle. To this point our Buffon models have included only those situations where the needle was not larger than the shortest distance between grid lines. When $l > d$, there can be multiple intersections. Problems connected with the probability of a specific number of crossings or moments of the distribution of a number of crossings arise.

First we will review this situation within the context of the prior section on statistical estimation of the value of π. Mantel (1953) developed a statistical estimator in the following way. We now view the expected number of intersections for a double grid system (equally unit-spaced) which Mantel writes as

$$E = \frac{4l}{\pi}. \tag{1.44}$$

No basis for this statement is given by Mantel and we shortly present a more general development by Morton (1966) from which the expectation statement can be derived. Assuming n throws of the needle we can get say c_i intersections at the ith fall; $i = 1, 2, \cdots, n$ and write

$$\hat{\pi} = \frac{4l}{\bar{c}} \tag{1.45}$$

as an estimation of

$$\pi = \frac{4l}{E} \tag{1.46}$$

where $\bar{c}$ is the average number of intersections per fall. By the delta method we can get

$$E(E - \bar{c})^2 \cong E(\pi - \hat{\pi})^2 \frac{16l^2}{\pi^4} \tag{1.47}$$

and

$$\text{AVar } \hat{\pi} = \frac{\pi^4}{16l^2} \frac{\sigma_c^2}{n} \tag{1.48}$$

where σ_c^2 is the variance of the number of intersections obtained at the fall of the needle.

Theoretical evaluation of σ_c for large l is of interest. Let $l \gg 1$ so that certain marginal effects can be disregarded. These marginal effects arise from the actual location of the end of the line within the squares in which it falls and they would slightly increase the value of σ_c^2 over what is now developed here but would have no effect on $E(c)$.

For any given angle θ at which the line of length l falls there would be $l \sin \theta$ intersections with vertical lines and $l \cos \theta$ intersections with horizontal lines. Thus the expected number of intersections is given by

$$E(c) = \frac{2}{\pi} \int_0^{\pi/2} l(\sin \theta + \cos \theta)\, d\theta = \frac{4l}{\pi}. \tag{1.49}$$

The expected square of the number of intersections is

$$\begin{aligned} E(c^2) &= \frac{2}{\pi} \int_0^{\pi/2} l^2(\sin \theta + \cos \theta)^2\, d\theta, \\ E(c^2) &= \frac{2}{\pi} \int_0^{\pi/2} l^2(1 + \sin 2\theta)\, d\theta, \\ E(c^2) &= l^2\left(1 + \frac{2}{\pi}\right). \end{aligned} \tag{1.50}$$

Thus

$$\begin{aligned} \sigma_c^2 &= Ec^2 - [E(c)]^2 = l^2\left(1 + \frac{2}{\pi} - \frac{16}{\pi^2}\right), \\ \sigma_{\hat{\pi}} &\cong \pi\sqrt{(\pi^2 + 2\pi - 16)/(16n)} \quad \textit{for large } l. \end{aligned} \tag{1.51}$$

Here

$$\sigma_{\hat{\pi}} = \sqrt{.0095/n}. \tag{1.52}$$

Previously for $l = d$ and in the single grid, $\sigma_{\hat{\pi}} = \sqrt{.5708/n}$. Thus the precision in estimating π from the double grid with a long needle is about 60 times as good as the former precision, or equivalently the information in one fall of the long needle here is about the same as in 60 falls of the needle in the original Buffon needle problem when the length of the needle is equal to the distance between the parallel lines.

In the Cartesian grid development we obtain

$$\frac{\sigma_c^2}{l^2} = 1 + \frac{2}{\pi} - \frac{16}{\pi^2}. \tag{1.53}$$

This suggests that an estimate of π can be made from the variation in number of intersections from fall to fall. Let

$$V = \frac{\hat{\sigma}_c^2}{l^2} \tag{1.54}$$

where $\hat{\sigma}_c$ is the *sample* standard deviation of intersections per fall. Then an estimate of π can be obtained from the solution to

$$(1-V)\pi^2+2\pi-16=0,$$

namely

$$\hat{\pi}=\frac{-1+\sqrt{1+16(1-V)}}{1-V}.$$

Let us now examine the sampling variation of $\hat{\pi}$. For any sample, V must be between 0 (all N falls give the same number of intersections) and the value of V obtained when half the lines have the minimum number of intersections and the other half the maximum number of intersections. The latter will occur when half the falls are parallel or perpendicular to the Cartesian grid system and the other half fall at an angle $\pi/4$ or $3\pi/4$ to the fixed grid. In this case, half of the times we will get l intersections and the other half of the times we will get $2l/\sqrt{2}$ intersections (each 45° line will have $l/\sqrt{2}$ intersections with the horizontal line and $l/\sqrt{2}$ intersections with the vertical lines). Thus we have

$$\sum_{i=1}^{n} c_i=\frac{n}{2}l+\frac{n}{2}\left(\frac{2l}{\sqrt{2}}\right)=\frac{n}{2}l(1+\sqrt{2}),$$

$$\sum_{i=1}^{n} c_i^2=\frac{n}{2}l^2+\frac{n}{2}(2l^2)=\frac{3n}{2}l^2,$$

$$V=\frac{\hat{\sigma}_c^2}{l^2}=\frac{[\sum c_i^2-(\sum c_i)^2/n]/n}{l^2}=\frac{3}{2}-\frac{1}{4}[3+2\sqrt{2}],$$

$$V=\frac{3-2\sqrt{2}}{4}.$$

Thus for $V=0$, $\hat{\pi}=3.1231$, and for $V=(3-2\sqrt{2})/4$, $\hat{\pi}=3.1752$.

This demonstrates that π must be between 3.1231 and 3.1752 without any experimentation and suggests that an actual simulation will give very satisfactory estimates. Mantel proceeds after these developments to give the results of a simulation to estimate π by each of the three methods. Each estimate is based on 101 falls and 90% confidence interval limits are given:

	Estimate
Buffon needle case, $l=1$	2.75–3.53
Double grid system	
Mean number of intersections	3.09–3.19
Variation in number of intersections	3.138–3.146

It would be interesting to see a development which combined the ideas of Mantel with those of Perlman and Wichura.

In a very recent paper, Diaconis (1976) has investigated several aspects of the single grid Buffon problem with a long needle. He derives the distribution of the

number of intersections and approximate moments for large l. The distribution is shown to converge weakly to an arc sin law as $l/d \to \infty$.

When $l > d$, the probability $p(i)$ of exactly i intersections is given as follows where the range of the number of intersections is 0 to $[a]+1$, $a = l/d$ and $[x]$ denotes the greatest integer less than or equal to x. Also let the angles θ_i $(0 \leqq \theta_i \leqq \pi/2)$ be determined by $\cos \theta_i = i/a$ and let $\delta_i = (2a \sin \theta_i/\pi) - (2i\theta_i/\pi)$. Then for $[a] \geqq 2$

$$p(0) = \delta_1 + 1 - \left(\frac{2a}{\pi}\right),$$

(1.55) $$p(i) = \delta_{i-1} + \delta_{i+1} - 2\delta_i \quad \text{for } 1 \leqq i \leqq M-2,$$

$$p(M-1) = \delta_{M-2} - 2\delta_{M-1}; \qquad p(M) = \delta_{M-1}.$$

For $[a] = 1$, the results for $p(0)$ and $p(M)$ above hold, and

(1.56) $$p(1) = \left(\frac{4\theta_1}{\pi}\right) + \left(\frac{2a}{\pi}\right) - (4a \sin \theta_1/\pi).$$

We can also write the distribution function in simple form as follows

$$p(\text{number of crossings} \leqq i) = F(i) = 1 - (\delta_i - \delta_{i+1}) \qquad i = 0, 1, \cdots, M-1.$$

Diaconis shows also that as $a = l/d \to \infty$, $k \geqq 1$

(1.57) $$u_k = \sum_{i=0}^{M} i^k p_i = c_k a^k + O(a^{k-3/2})$$

where

$$c_k = \{\Gamma[\tfrac{1}{2}(k+1)]/\Gamma[\tfrac{1}{2}(k+2)]\}\sqrt{\pi}.$$

As $a \to \infty$, the moments of I/a converge to c_k where I is the random variable describing the number of crossings of the needle. However the arc sin distribution has moments c_k and since the range of all concerned distributions is the unit interval, (I/a) converges in distribution to

(1.58) $$f(x) = \begin{cases} \left(\dfrac{2}{\pi}\right)\dfrac{1}{(1-x^2)^{1/2}}, & 0 \leqq x \leqq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Kendall and Moran (1963, pp. 73–74) list some values of $p(i)$ and Feller (1971, p. 527) refers to the arc sin density for this problem.

General extension of Buffon problem. A rather general extension of the Buffon problem is provided by Morton (1966). Morton proves an important and remarkable result on the expected number of intersections and distribution of angles of intersections for a random set of rectifiable curves in the plane.

Let $\mathscr{C}$ be a set of rectifiable curves $C_1, \cdots, C_m$ and $\mathscr{D}$ a set of rectifiable curves $D_1, \cdots, D_N$ of total length $l_{\mathscr{C}}$ and $l_{\mathscr{D}}$ respectively. The curves fall

"randomly" over the area A and we assume only that (i) the arrangement of the two groups of curves on the area A must be independent of each other although the individual curves of a group may have a systematic arrangement relative to each other, (ii) the arrangement of at least one of the groups on A must be random with the random mechanism operating such that the probability of a specified point on a curve falling into a subarea of A is proportional to its area and the segment may assume any angle relative to some base line with equal probability.

Then the expected number of intersections of the two groups is

$$E=\frac{2l_{\mathscr{C}}l_{\mathscr{D}}}{\pi A}. \tag{1.59}$$

Furthermore, the density of the intersection angle between the tangents of any two of the C_i, D_j is

$$\tfrac{1}{2}\sin\theta\,d\theta, \qquad 0\leqq\theta\leqq\pi. \tag{1.60}$$

We now sketch Morton's proof. Partition each C_α with points $C_{\alpha 0}, \cdots, C_{\alpha\gamma_\alpha}$ so that for $i=1,2,\cdots,\gamma_\alpha$, $C_{\alpha i}$ can be jointed to $C_{\alpha i-1}$ by a line segment of length δl. Similarly partition D_β. (δl is small enough so that end effects can be ignored. See Fig. 1.2.)

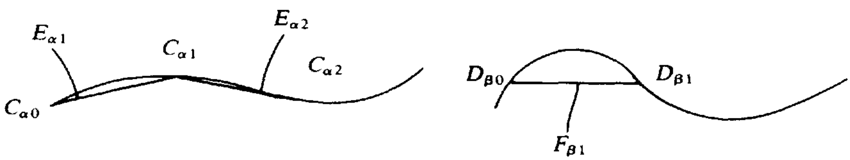


FIG. 1.2

Label the line segments $E_{\alpha 1}$ etc., $F_{\beta 1}$ etc. Let

(1.61) $\theta_{\beta_j}^{\alpha_i}$ be the angle between $E_{\alpha i}$, $F_{\beta j}$ or their extensions.

In order that $E_{\alpha i}$ and $F_{\beta j}$ intersect, the midpoint of $F_{\beta j}$ must lie within a rhombus with sides parallel to $E_{\alpha i}$ and $F_{\beta j}$ and of length δl (see Fig. 1.3) and such that $E_{\alpha i}$ joins the midpoints of the sides parallel to $F_{\beta j}$. This is under the assumption that $\theta_{\beta_j}^{\alpha_i}$ is given.

The area of the rhombus is its altitude times the length of a side, that is,

$$[(\delta l)\sin\theta_{\beta_j}^{\alpha_i}](\delta l). \tag{1.62}$$

Therefore via assumption (ii)

$$P[E_{\alpha i}, F_{\beta j} \text{ intersect} \mid \theta_{\beta_j}^{\alpha_i}]=\frac{(\delta l)^2\sin\theta_{\beta_j}^{\alpha_i}}{A}. \tag{1.63}$$

Let

$$I(\alpha_i,\beta_j)=\begin{cases}1, & E_{\alpha_i}, F_{\beta_j} \text{ intersection},\\ 0 & \text{otherwise}.\end{cases}$$

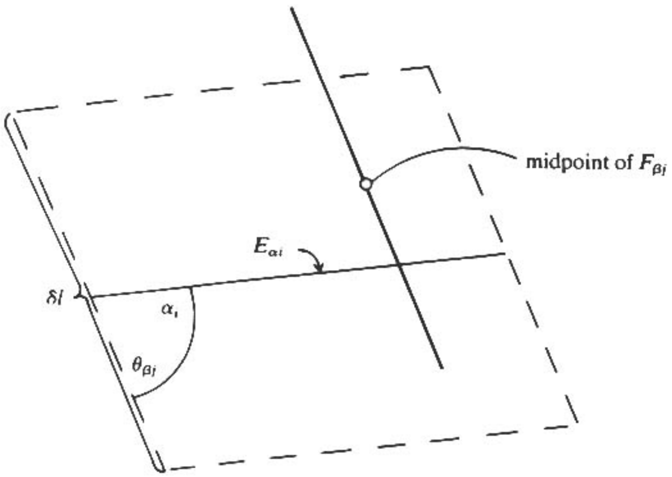


FIG. 1.3

Then the left hand side is $P[I(\alpha_i, \beta_j)=1\,|\,\theta_{\beta_j}^{\alpha_i}]$. Thus via Bayes rule,

$$P[\theta_{\beta_j}^{\alpha_i}|I(\alpha_i, \beta_j)]=1=\frac{P[I(\alpha_i, \beta_j)=1\,|\,\theta_{\beta_j}^{\alpha_i}]P(\theta_{\beta_j}^{\alpha_i})}{\int P[I(\alpha_i, \beta_j)=1\,|\,\theta_{\beta_j}^{\alpha_i}]P(\theta_{\beta_j}^{\alpha_i})\,d\theta_{\beta_j}^{\alpha_i}}. \tag{1.64}$$

From (ii) we have $P(\theta_{\beta_j}^{\alpha_i})$ is uniform on $[0, \pi]$. Therefore,

$$P[\theta_{\beta_j}^{\alpha_i}|\,I(\alpha_i, \beta_j)=1]=\frac{(\delta l)^2\sin\theta_{\beta_j}^{\alpha_i}(1/(\pi a))}{\int_0^\pi(\delta l)^2\sin\theta_{\beta_j}^{\alpha_i}(1/(\pi A))\,d\theta_{\beta_j}^{\alpha_i}}=\frac{1}{2}\sin\theta_{\beta_j}^{\alpha_i}. \tag{1.65}$$

If in the limit as $\delta l\to 0$, E_{α_i} and F_{β_j} do in fact intersect, then $\theta_{\beta_j}^{\alpha_i}\to\theta$, the angle between the tangents of $\mathscr{C}$ and $\mathscr{D}$ measured at the point of intersection of the segment $C_{\alpha,i-1}$, $C_{\alpha i}$ with the segment $D_{\beta,j-1}$, $D_{\beta j}$ and thus the limting density element is

$$\tfrac{1}{2}\sin\theta, \qquad 0\leqq\theta\leqq\pi.$$

This result was developed somewhat earlier by Wolfowitz (1949) who addressed his paper only to intersection angles.

Furthermore, the probability,

$$\begin{aligned} P[E_{\alpha_i}, F_{\beta_j}\text{ intersect}] &= \int_0^\pi P[I(\alpha_i, \beta_j)=1\,|\,\theta_{\beta_j}^{\alpha_i}]P(\theta_{\beta_j}^{\alpha_i})\,d\theta_{\beta_j}^{\alpha_i} \\ &= \int_0^\pi\frac{(\delta l)^2\sin\theta_{\beta_j}^{\alpha_i}}{A}\,\frac{1}{\pi}\,d\theta_{\beta_j}^{\alpha_i} \\ &= \frac{2(\delta l)^2}{\pi A}. \end{aligned} \tag{1.66}$$

Thus finally the expected number of intersections is

$$E = \frac{2(\delta l)^2}{\pi A} \lim_{\delta l \to 0} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \gamma_\alpha \rho_\beta,$$

$$\begin{aligned} &\gamma_\alpha = \text{number of segments that approximate } C_\alpha, \\ &\rho_\beta = \text{number of segments that approximate } D_\beta, \end{aligned} \tag{1.67a}$$

$$E = \frac{2}{\pi A} \sum_{\alpha=1}^{m} \delta l \gamma_\alpha \sum_{\beta=1}^{n} \delta l \rho_\beta$$

and

$$\begin{aligned} &\lim_{\delta l \to 0} \sum_{\alpha=1}^{m} \delta l \gamma_\alpha = \sum \text{length of curves in set } \mathscr{C} = l_{\mathscr{C}}, \\ &\lim_{\delta l \to 0} \sum_{\beta=1}^{n} \delta l \rho_\beta = \sum \text{length of curves in set } \mathscr{D} = l_{\mathscr{D}}. \end{aligned} \tag{1.67b}$$

The relationship between the expected number of intersections of a fixed group of line segments of total length L_1 with a group of line segments of total length L_2 that fall randomly over an area A that encompases the fixed group of line segments has been demonstrated to be

$$E = \frac{2L_1L_2}{\pi A}.$$

This relationship was developed for segments which are not necessarily linear. Let us review the conditions and consider only linear segments:

(i) the arrangement of the two groups of line segments on the area A must be independent of each other, but the individual line segments of a group may have a systematic arrangement relative to each other;

(ii) the arrangement of at least one of the two groups of line segments on the area A must be at random, and the randomness must be such that the probability of a specified point on a line segment falling into a subarea of A is proportional to its area and the segment may assume any angle relative to some base line with equal probability.

In the Buffon needle problem, the expected number of intersections is equivalent to the probability that the needle of length $l \leqq 1$ (the distance between the parallel lines is taken as unity without loss of generality) intersects the parallel lines since an intersection, if it occurs, can be with only one parallel line. Thus we have from the result on expected number of intersections,

$$E = P\{\text{needle intersects line}\} = \frac{2l}{\pi}$$

since the total length of the parallel lines, namely L_2, is A. (Border effects can be made trivial by taking areas with dimensions which are longer relative to the distance between parallel lines).

If we now consider the double grid equally unit-spaced model and $l \leqq 1$, then

$$E = \frac{4l}{\pi} \tag{1.68}$$

and this is the assertion used by Mantel in his long needle experimental design.

It is difficult to leave the Buffon model and its generalizations and ramifications. For example, it can be shown that if a region bounded by a convex curve whose dimensions are such that the region cannot intersect two of the lines of the grid simultaneously is thrown "randomly" on the grid, the probability that the convex curve will intersect a line is given by

$$p = \frac{L}{\pi d} \tag{1.69}$$

where L is the length of the convex curve. If we view a straight line as a degenerate convex curve, that is, approximated by a rectangle with infinitesimal width and length l and thus with perimeter $2l$ we obtain Buffon's result $p = 2l/(\pi d)$. However, it is interesting that a proof of the more general statement can be obtained from Buffon's result as follows.

Suppose we can prove the result for any convex polygon of diameter size less than d. As the number of sides increases we can approximate any convex curve and hence the result will follow.

Therefore, consider an n-sided polygon with sides $a_1, \cdots, a_n$ and diameter less than d. See Fig. 1.4. Now if there is an intersection with a line of the grid two sides must cross the line (the measure of the set of outcomes with a corner of the polygon touching the line of the grid is zero).

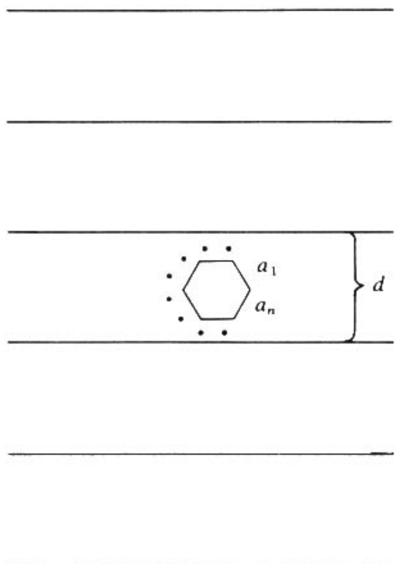


FIG. 1.4a

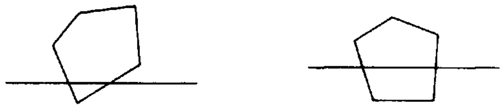


FIG. 1.4b

Let

$$p(a_i) = \text{probability side } a_i \text{ intersects the grid},$$

$$p(a_ia_j) = \text{probability sides } a_i \text{ and } a_j \text{ intersect the grid}.$$

Employing the above observation along with the fact that at most one intersection with the grid is possible, due to the size of the polygon, we have

$$P(\text{intersection}) = \sum_{i<j} p(a_ia_j). \tag{1.70}$$

But

$$p(a_i) = \sum_{\substack{j=1 \\ j\neq i}} p(a_ia_j).$$

Therefore

$$\begin{aligned} 2P(\text{intersection}) &= 2\sum_{i<j} p(a_ia_j) \\ &= \sum_{i=1} \sum_{\substack{j=1 \\ j\neq i}} p(a_ia_j) \\ &= \sum_{i=1}^{n} p(a_i). \end{aligned} \tag{1.71}$$

Each side of the polygon can be treated as the needle in the Buffon problem. Picking a side determines the location of the polygon and with sides picked equiprobably we get:

$$p(a_i) = \frac{2l_i}{\pi d} \quad \text{where } l_i = \text{length of } a_i.$$

Thus

$$P(\text{intersection}) = \sum \frac{l_i}{\pi d} = \frac{L}{\pi d} \tag{1.72}$$

where L is the perimeter of the polygon.

Buffon problem in three dimensions and an application. The three dimensional situation motivating the forthcoming discussion is the estimation of the volume-surface ratio of cells or nuclei. This was explored and presented in two interesting papers by Chalkley, Cornfield, and Park (1949) and Cornfield and

Chalkley (1951). This index might serve to delineate some pathology, for example, cancerous versus noncancerous cells. The method proposed in the papers takes as its point of departure a remarkable result by Crofton that we discuss later which shows that if a line is repeatedly placed at random over a plane containing a closed figure, the average length of the chord intersected by the figure will be

$$E(C)=\pi\frac{\text{area}}{\text{perimeter}} \tag{1.73}$$

no matter what the shape of the figure, as long as its boundary is a convex curve.

Before this result can be employed for estimating volume-surface area ratios it requires modifications:

a) it must be made applicable to line segments,
b) a simple procedure for measuring chord length must be found,
c) it must be extended to cover reentrant figures,
d) it must be extended to provide an estimate not only of the area-perimeter ratio in the focal plane under observation but of the volume-surface ratio in the three dimensions of which the observed focal plane is a two-dimensional representation.

The first three of these modifications are accomplished by a single device. Consider a line of finite length, say r, and count the number of times each of the two end points fall in the interior of a plane figure and denote this by h for hits. Denote the number of times the line intersects the perimeter of the figure by c for cuts. Then in a *very large* number of throws we shall find

$$r\cdot\frac{E(h)}{E(c)}=\pi\frac{\text{area}}{\text{perimeter}} \tag{1.74}$$

for all closed figures including reentrant ones.

The fourth and important modification is provided by a mathematical result. When a line of length r is placed at random in three dimensional space containing a closed figure, for a very large number of throws we may write

$$r\cdot\frac{E(h)}{E(c)}=4\frac{\text{volume}}{\text{surface area}}. \tag{1.75}$$

This result will satisfy condition d) since placing a line at random in three dimensions can be shown to be formally equivalent to placing a plane at random in three dimensions and placing the line at random on the resulting two-dimensional plane section.

If the space contains a series of figures of different volumes and surfaces, then

$$r\cdot\frac{E(h)}{E(c)}=4\frac{\text{sum of volumes}}{\text{sum of surface areas}}. \tag{1.76}$$

As in two dimensions, this result applies whatever the shape of the figure, and covers reentrant as well as convex figures.

Basic to the mathematical result is a particular definition of randomness and this is where we hark back to the Buffon needle problem development. Other definitions are possible and these lead to different results. Therefore a test of the method is necessary even though it was verified empirically in the Buffon problem. This is discussed in the papers by Chalkley et al. (1949) and Cornfield and Chalkley (1951).

Assume that in throwing a line of length r at random

a) the probability that one end point, P, will take on any position in the space is uniformly distributed over the space,
b) the probability that the other end point, P', will take any position on the surface of the sphere with center at P and radius r is uniformly distributed over the surface.

This is a generalization of the assumptions usually made in geometrical probability in two dimensions, for example, in Buffon's needle problem. In the first of the two papers we have mentioned, evidence was presented that this appears to provide a satisfactory description of the physical process of throwing a line segment at random.

First we show that

$$E(h)=\frac{2Vn}{X} \tag{1.77}$$

where X is the volume of the space in which the line segment falls, n is the number of throws, and V is the volume in which a hit occurs.

Let us begin with the two-dimensional problem (see Fig. 1.5).

The endpoint, P, of the line segment will occupy with uniform probability all positions in the square of side $X^{1/2}$. Since the angle that the line segment makes with the X axis, θ, varies from 0 to π, the point P' will occupy all but some corner positions in the square of side $X^{1/2}+2r$.

If we consider fixed values of y and θ, the probability that the point P will fall inside the figure is

$$\frac{w(y)}{X^{1/2}} \tag{1.78}$$

where $w(y)$ is the width of the figure at ordinate y. Similarly the probability that for fixed values of y and θ the point P' will fall inside the figure is

$$\frac{w(y,\theta)}{X^{1/2}}. \tag{1.79}$$

If we now let y vary from 0 to $X^{1/2}$ but hold θ fixed, we have

$$\begin{aligned} \Pr(P\in F)&=\frac{1}{X}\int_0^{X^{1/2}} w(y)\,dy, \\ \Pr(P'\in F)&=\frac{1}{X}\int_0^{X^{1/2}} w(y,\theta)\,dy. \end{aligned} \tag{1.80}$$

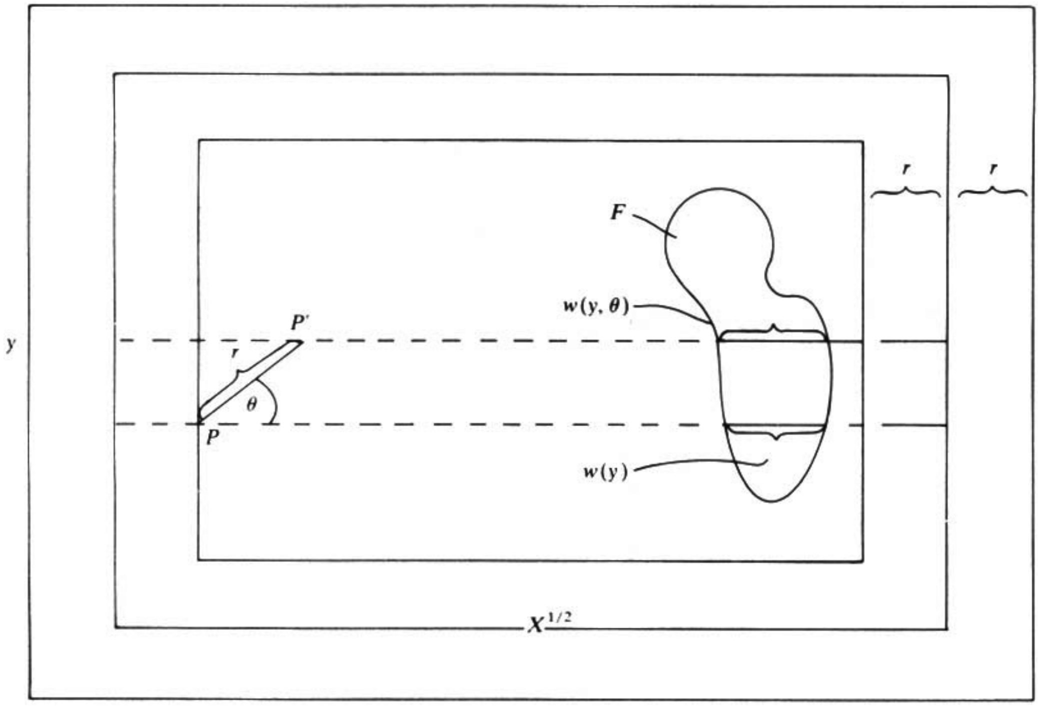


FIG. 1.5

Each integral obviously is an expression for the area of the figure, A, and is consequently independent of θ. In n throws, therefore, the expected number of hits by either the point P or the point P' is $n \cdot (A/X)$. The expected number of hits counting both endpoints is

$$E(h) = 2 \cdot n \cdot \frac{A}{X} \quad \text{for 2 dimensions.} \tag{1.81}$$

A similar proof in three dimensions, with a similar diagram, will yield with X the volume of the cube of side $X^{1/3}$,

$$\begin{aligned} &\Pr(P \in F) = \frac{w(y, z)}{X^{1/3}} \qquad \text{for } y, z \text{ fixed,} \\ &\Pr(P' \in F) = \frac{w(y, z, \theta, \varphi)}{X^{1/3}} \quad \text{for } y, z, \theta, \varphi \text{ fixed,} \end{aligned} \tag{1.82}$$

(where Pr stands for probability). Therefore

$$\Pr(P \in F) = \int_0^{X^{1/3}} \int_0^{X^{1/3}} \frac{1}{X^{1/3}} \cdot \frac{1}{X^{1/3}} \frac{w(y, z)}{X^{1/3}} \, dy \, dz = \frac{V}{X} \tag{1.83}$$

and

$$\Pr(P' \in F) = \int_0^{X^{1/3}} \int_0^{X^{1/3}} \frac{1}{X^{1/3}} \frac{1}{X^{1/3}} \frac{w(y, z, \theta, \varphi)}{X^{1/3}}\, dy\, dz = \frac{V}{X}$$

and so again in n throws the expected number of hits

(1.84) $$E(h) = n\frac{V}{X} + n\frac{V}{X} = \frac{2nV}{X}.$$

Let us now calculate $E(c)$. Assume without loss of generality that F is composed of m quadrilaterals of area $S_1, S_2, \cdots, S_m$, i.e., $S =$ surface of $F = \sum_{i=1}^m S_i$. If $c_i =$ number of cuts on ith quadrilateral

$$E(c) = \sum_{i=1}^m E(c_i).$$

In calculating $E(c_i)$ we note that since a straight line and a quadrilateral can either intersect at only one point or not intersect, see Fig. 1.6, $E(c_i) =$ probability of an intersection multiplied by the number of throws.

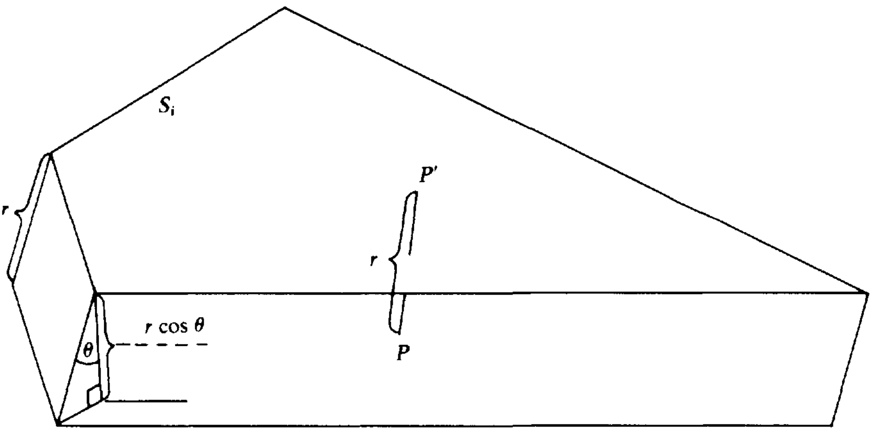


FIG. 1.6

If we consider the ith quadrilateral with surface area S_i, the probability that a random line of length r will intersect it is the probability that the endpoint P falls within the above parallelepiped. Since the volume of the parallelepiped is $S_i r \cos\theta$ this probability is

(1.85) $$\frac{S_i r \cos\theta}{X}, \quad \text{given } \theta$$

and thus the unconditional probability is given by

$$\int_0^{\pi/2} \frac{S_i r \cos\theta f(\theta)}{X} \, d\theta. \tag{1.86}$$

We will show that $f(\theta) = \sin\theta$. Assume for the moment that this is so; then

$$\Pr(P \in \text{Parallelepiped}) = \frac{S_i r}{X} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \frac{S_i r}{2X}. \tag{1.87}$$

Thus

$$E(c_i) = \frac{rn}{2X} S_i \quad \text{and} \quad E(c) = \sum \frac{rn}{2X} S_i = \frac{rnS}{2X}.$$

Thus

$$\frac{E(h)}{E(c)} = \frac{2n(V/X)}{rnS/(2X)} = \frac{4V}{rS}, \quad \text{or} \quad r \cdot \frac{E(h)}{E(c)} = 4\frac{V}{S}. \tag{1.88}$$

Now let us consider the probability distribution of θ. We seek the probability distribution of the angle θ formed by a random line with a fixed line under our assumption of randomness. To see that the density element is $\sin\theta \, d\theta$ let the fixed line be the radius of the sphere with center at P and radius r. (See Fig. 1.7.)

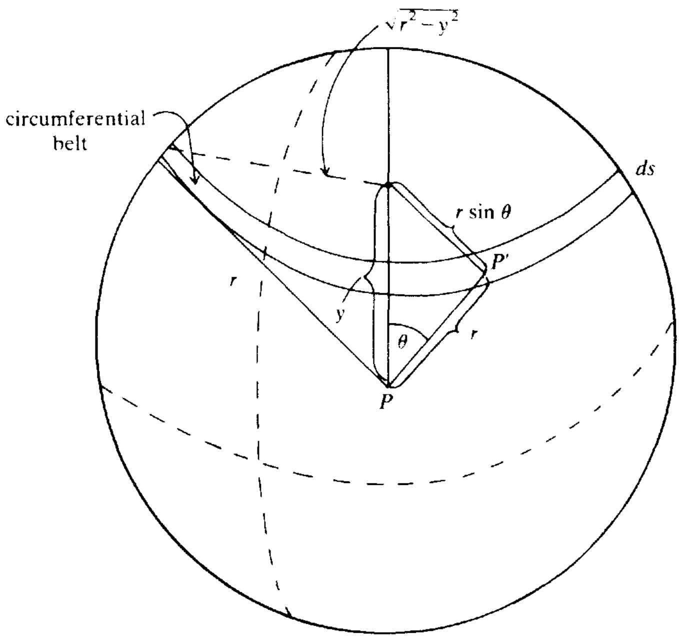


FIG. 1.7

The probability that the line with endpoints P and P' forms an angle θ with the radius of the sphere is the probability that the endpoint P' falls in the indicated circumferential belt or

$$\text{Probability} = \frac{\text{surface area of circumferential belt}}{\text{surface area of sphere with radius } r},$$

(1.89) or

$$\frac{2\pi(\sqrt{r^2 - y^2}) \cdot ds}{2\pi r^2} = \frac{\sin\theta\, ds}{r}$$

but $s = r\theta$ implies

$$ds = r\,d\theta \tag{1.90}$$

and thus the desired density element

$$f(\theta)\,d\theta = \sin\theta \frac{r\,d\theta}{r} = \sin\theta\,d\theta \quad \text{and} \quad f(\theta) = \sin\theta. \tag{1.91}$$

CHAPTER 2

Density and Measure for Random Geometric Elements

In our Buffon discussion we have referred to the random positioning of a line segment in the plane and in a brief way to the similar situation for a line in the plane. For the solutions we have developed for probabilities or expectations of events, we have employed $dp\,d\theta$ as the appropriate density and $\int_A dp\,d\theta$ as the measure over some region A where p and θ are the coordinates of the line in normal form. The density $dp\,d\theta$ and measure based on it for events regarding lines in the plane will lead to probability and expectation statements that are invariant under the group of rigid motions in the plane, that is, translation and rotation. For some subsequent problems on lines and line segments it will be found that invariance under rotation is too demanding and we will relax this condition.

For the present we turn our attention to the more general question of density and measure for random geometric elements that leave probability and expectation statements invariant under translation and rotation. To do this we borrow heavily from Santaló (1953), (1976). Remarkable results due to Crofton (1885) fall out of the analysis as the principal applications for this chapter.

When a point in the plane is described by its Cartesian coordinates x, y from an arbitrary origin, the appropriate measure for the set of points in a region A is given by $\int_A dx\,dy$ and this measure is invariant under translation and rotation. The appropriate density is $dx\,dy$ except for a constant factor and this can always be treated as unity. All this is accomplished by transforming the point x', y' to x, y by the group of rigid motions and noting that the Jacobian of the transformation is unity.

If we wish to transform the point (x, y) to (u, v) where $x = x(u, v)$ and $y = y(u, v)$ and then get $dx\,dy$ in terms of $du\,dv$ we have to be somewhat formal and employ the exterior multiplication of differential forms or wedge product notions to accomplish this; see for example Flanders (1963). We will employ square brackets to indicate this operation, for example, $dP = [dx\,dy]$ and the rules are:

1) The product is equal to zero if any two factors are equal.

2) The product is unchanged by an even permutation of the factors and is multiplied by (-1) for an odd permutation of factors.

Now we return to our example where

$$x = x(u, v), \qquad y = y(u, v).$$

Then

$$(2.1)\qquad dx=\frac{\partial x(u,v)}{\partial u}du+\frac{\partial x(u,v)}{\partial v}dv,\qquad dy=\frac{\partial y(u,v)}{\partial u}du+\frac{\partial y(u,v)}{\partial v}dv$$

and

$$[dx\,dy]=\frac{\partial x(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v}[du\,dv]+\frac{\partial x(u,v)}{\partial v}\frac{\partial y(u,v)}{\partial u}[dv\,du]$$

$$+\text{two terms equal to zero,}$$

$$(2.2)\qquad [dx\,dy]=\left\{\frac{\partial x(u,v)}{\partial u}\frac{\partial y(u,v)}{\partial v}-\frac{\partial x(u,v)}{\partial v}\frac{\partial y(u,v)}{\partial u}\right\}[du\,dv].$$

That is, we get the usual Jacobian.

Let us look into an application of these ideas. Consider a convex plane curve K with a tangent at every point. Let O be a point inside K. (See Fig. 2.1.)

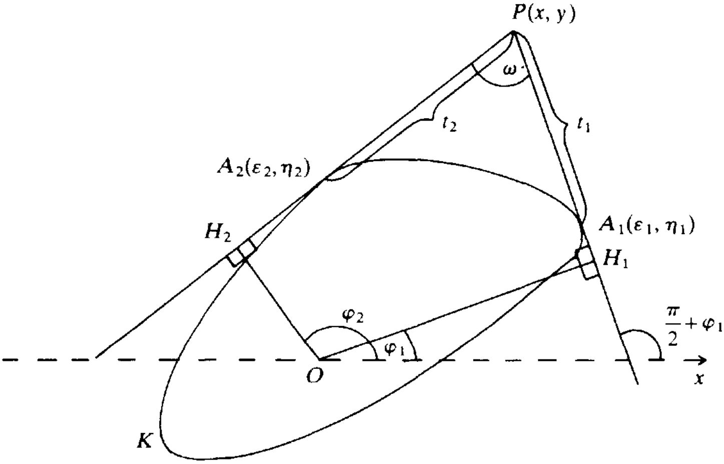


FIG. 2.1

From each point P exterior to K there can be drawn two tangents to K, say PA_1 and PA_2. To each of these tangents corresponds an angle φ_1 and φ_2 formed by the perpendiculars OH_1 and OH_2 with the fixed direction Ox. Conversely, the two angles φ_1, φ_2 determine the point $P(x, y)$. We wish now to express the density dP in terms of the coordinates φ_1, φ_2.

Let ε_1, η_1 be the coordinates of the point of tangency A_1 and x, y the coordinates of P. The equation of the straight line PA_1 is

$$(2.3)\qquad (x-\varepsilon_1)\cos\varphi_1+(y-\eta_1)\sin\varphi_1=0.$$

Likewise, the equation of the second tangent PA_2 is

$$(2.4)\qquad (x-\varepsilon_2)\cos\varphi_2+(y-\eta_2)\sin\varphi_2=0.$$

By differentiation we get

$$
\begin{aligned}
&\cos\varphi_1\, dx + \sin\varphi_1\, dy = [(x-\varepsilon_1)\sin\varphi_1 - (y-\eta_1)\cos\varphi_1]\, d\varphi_1,\\
&\cos\varphi_2\, dx + \sin\varphi_2\, dy = [(x-\varepsilon_2)\sin\varphi_2 - (y-\eta_2)\cos\varphi_2]\, d\varphi_2.
\end{aligned} \tag{2.5}
$$

Since PA_1 is tangent to K at A_1, $d\eta_1/d\varepsilon_1$ is equal to the slope of the line PA_1, namely $\tan(\pi/2+\varphi_1) = -\cot\varphi_1$, or $\cos\varphi_1\, d\varepsilon_1 + \sin\varphi_1\, d\eta_1 = 0$. Equivalently we get $\cos\varphi_2\, d\varepsilon_2 + \sin\varphi_2\, d\eta_2 = 0$.

Now recall that $(x-\varepsilon_1)\cos\varphi_1 + (y-\eta_1)\sin\varphi_1 = 0$ and square both sides of the equation. We get

$$
(x-\varepsilon_1)^2\cos^2\varphi_1 + (y-\eta_1)^2\sin^2\varphi_1 + 2(x-\varepsilon_1)(y-\eta_1)\cos\varphi_1\sin\varphi_1 = 0. \tag{2.6}
$$

Note that

$$
\begin{aligned}
[(x-\varepsilon_1)\sin\varphi_1 + (y-\eta_1)\cos\varphi_1]^2 &= (x-\varepsilon_1)^2\sin^2\varphi_1 + (y-\eta_1)^2\cos^2\varphi_1\\
&\quad - 2(x-\varepsilon_1)(y-\eta_1)\cos\varphi_1\sin\varphi_1\\
&= (x-\varepsilon_1)^2\sin^2\varphi_1 + (y-\eta_1)^2\cos^2\varphi_1\\
&\quad - [-(x-\varepsilon_1)^2\cos^2\varphi_1 - (y-\eta_1)^2\sin^2\varphi_1]\\
&= (x-\varepsilon_1)^2 + (y-\eta_1)^2 = t_1^2.
\end{aligned}
$$

Thus

$$
(x-\varepsilon_1)\sin\varphi_1 - (y-\eta_1)\cos\varphi_1 = t_1 \tag{2.7}
$$

and similarly

$$
(x-\varepsilon_2)\sin\varphi_2 - (y-\eta_2)\cos\varphi_2 = t_2.
$$

By substitution we get

$$
\cos\varphi_1\, dx + \sin\varphi_1\, dy - t_1\, d\varphi_1 = 0, \quad \cos\varphi_2\, dx + \sin\varphi_2\, dy - t_2\, d\varphi_2 = 0. \tag{2.8}
$$

But by *exterior multiplication* we obtain

$$
\begin{aligned}
&(\cos\varphi_2\sin\varphi_1 - \sin\varphi_2\cos\varphi_1)[dx\, dy] = t_1 t_2 [d\varphi_1\, d\varphi_2],\\
&\sin(\varphi_2-\varphi_1)[dx\, dy] = t_1 t_2 [d\varphi_1\, d\varphi_2].
\end{aligned} \tag{2.9}
$$

Furthermore, $\varphi_2 - \varphi_1 = \pi - \omega$ where ω is the angle A_1PA_2 formed by the tangents from P. Thus we have

$$
dP = [dx\, dy] = \frac{t_1 t_2}{\sin\omega}[d\varphi_1\, d\varphi_2] \quad \text{or} \quad \frac{\sin\omega}{t_1 t_2}\, dP = [d\varphi_1\, d\varphi_2]. \tag{2.10}
$$

Integrate both sides of this equality over all possible different values of the variables—P can vary over all points exterior to K; φ_1, φ_2 can vary from 0 to 2π. However, if in each position we permute φ_1 and φ_2 we get the *same* point P.

Consequently, to count each point P only once we must divide by two. Therefore,

$$\int_{P \text{ ext to } K} \frac{\sin \omega}{t_1 t_2} dP = \frac{1}{2}(2\pi)(2\pi) = 2\pi^2. \tag{2.11}$$

This integral formula is due to Crofton. Note that the right hand member does not depend upon the convex curve K.

If K has a continuous radius of curvature ρ_1 we can set $\rho_1\, d\varphi_1 = ds_1$, $\rho_2\, d\varphi_2 = ds_2$ where ds_1, ds_2 are arc lengths of K at A_1, A_2. Then we may write

$$\frac{\sin \omega}{t_1 t_2} \rho_1 \rho_2\, dP = [ds_1\, ds_2] \tag{2.12}$$

since $\rho_1\rho_2\, d\varphi_1\, d\varphi_2 = ds_1\, ds_2$ and $[d\varphi_1\, d\varphi_2] = (\sin \omega/(t_1 t_2))\, dP$.

Now integrate both sides over all possible values of the variables

$$\int_{P \text{ ext to } K} \frac{\sin \omega}{t_1 t_2} \rho_1 \rho_2\, dP = \frac{1}{2} L^2 \tag{2.13}$$

where L is the length of K.

It can also be shown (Santaló (1953)) that

$$\int_{P \text{ ext to } K} \frac{\sin \omega}{t_1 t_2} (\rho_1 + \rho_2)\, dP = 2\pi L. \tag{2.14}$$

Straight lines in the plane. Represent any line G in the plane by its normal coordinates (p, θ), where p is the distance of the normal to the line from an origin and $0 \leqq \theta \leqq \pi$ is the angle formed by the normal to the line and the x-axis. The equation for G can be written as

$$x \cos \theta + y \sin \theta - p = 0. \tag{2.15}$$

The group of transformations with respect to which we seek an invariant measure is the same as for points in the plane, namely the group of rigid motions. From Santaló (1953), we get $dp\, d\theta$ as the invariant density and thus the measure $\mathrm{M}(A)$ of a set A of straight lines, $G(p, \theta) \in A$, in the plane is defined by

$$\mathrm{M}(A) = \int_A dp\, d\theta. \tag{2.16}$$

Up to a constant factor, this measure is the only one which is invariant under the group of motions in the plane. The differential form under the integral is called the density for straight lines and is represented by

$$dG = [dp\, d\theta]. \tag{2.17}$$

If we take for coordinates of G the coefficients u, v in its equation $ux + vy + 1 = 0$, it can be shown that

$$dG = \frac{[du\, dv]}{(u^2 + v^2)^{3/2}}. \tag{2.18}$$

This can be seen by noting that

$$(2.19)\qquad u=-\frac{\cos\theta}{p},\qquad v=-\frac{\sin\theta}{p}.$$

Suppose we represent G by its intercepts α, β on the coordinate axes. Then

$$(2.20)\qquad \alpha=-\frac{p}{\cos\theta},\qquad \beta=\frac{p}{\sin\theta},$$

and in similar fashion it can be shown

$$(2.21)\qquad dG=\frac{\alpha\beta[d\alpha\, d\beta]}{(\alpha^2+\beta^2)^{3/2}}.$$

In the Bertrand paradox, the probability that a random chord is larger than the side of an inscribed equilateral triangle is $\frac{1}{2}$ when the density is $dp\, d\theta$. This suggests that the other solutions of $\frac{1}{3}$ and $\frac{1}{4}$ are generated by random mechanisms that do not provide invariant measure under the group of rigid motions. Specifically we illustrate this as follows. Let the length of the chord depend on the distance from the center of a circle and not on the direction. Suppose that it has a fixed direction perpendicular to a given diameter of the circle and that its point of intersection with this diameter has a uniform distribution. For the chord to have a length greater than $\sqrt{3}$, the distance of the points of intersection from the center of the circle must be less than $\frac{1}{2}$. Thus the probability is $\frac{1}{2}$. Moreover, we can write in this case

$f(p\mid\theta)$ is uniformly distributed on the interval $(0, 1)$,

$f(\theta)$ is uniformly distributed on the interval $(0, 2\pi)$.

Therefore

$$(2.22)\quad f(p,\theta)\, dp\, d\theta=f(p\mid\theta)f(\theta)\, dp\, d\theta=1/(2\pi)\, dp\, d\theta\quad\text{or,}\quad f(p,\theta)=1/(2\pi),$$

and so the density is $dp\, d\theta$ up to a constant factor.

Measure and density for straight lines that intersect a curve provide interesting and remarkable results. Let C be a fixed curve composed of a finite number of connected arcs with a tangent at every point. Let C have length L and define C by $x=x(s)$, $y=y(s)$ where the parameter s is arc length.

In Fig. 2.2, G intersects C at (x, y) and forms an angle ω with the tangent at this point; τ is the angle the tangent makes with the x-axis. The length s corresponding to x and y along with the angle ω determine G uniquely for C and fixed origin O. Let us find dG in terms of s and ω, rather than p and θ as listed in the figure.

Clearly

$$(2.23)\qquad \theta+\pi/2+(\pi-\omega)+(\pi-\tau)=2\pi;$$

thus

$$\theta=\omega+\tau-\pi/2.$$

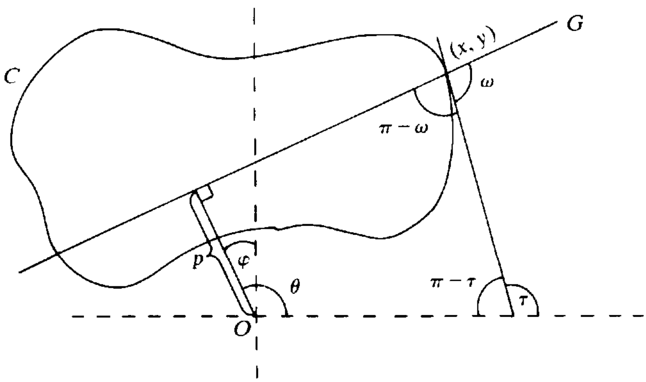


FIG. 2.2

Since $x, y \in G$

$$x \cos \theta + y \sin \theta - p = 0.$$

Therefore,

$$dp = \cos \theta \, dx + \sin \theta \, dy - (x \sin \theta - y \cos \theta) \, d\theta; \tag{2.24}$$

but $dx/ds = \cos \tau$, $dy/ds = \sin \tau$ yield

$$dp = \cos \theta \cos \tau \, ds + \sin \theta \sin \tau \, ds - (x \sin \theta - y \cos \theta) \, d\theta \tag{2.25}$$

so $dp = \cos(\theta - \tau) \, ds - (x \sin \theta - y \cos \theta) \, d\theta$.

Since τ is clearly a function of s alone $d\theta = d\omega + \tau'(s) \, ds$ is implied by $\theta = \omega + \tau - \pi/2$.

Then exterior multiplication yields

$$\begin{aligned} dG = [dp \, d\theta] &= \cos(\omega - \pi/2)[ds(d\omega + \tau' \, ds)] \\ &= \sin \omega [ds \, d\omega]. \end{aligned} \tag{2.26}$$

Through integration we have

$$\int n(p, \theta) \, dp \, d\theta = \int \sin \omega \, d\omega \, ds \tag{2.27}$$

where $n(p, \theta)$ is the number of intersections of the line (p, θ) with C. That is we must take into account the fact that every time a line G intersects C it counts into the left hand side. If G does not intersect C, $n(p, \theta) = 0$. Thus $\int n(p, \theta) \, dG = \int_0^L \int_0^\pi \sin \omega \, d\omega \, ds = 2L$. Furthermore if C is convex then

$$n(p, \theta) = \begin{cases} 0 & \text{if } G \text{ does not intersect } C, \\ 2 & \text{if } G \text{ does intersect } C \end{cases}$$

and thus

$$\int_{G \cap C} n(\rho, \theta) \, dG = 2L \quad \text{becomes} \quad \int_{G \cap C} 2 \, dG = 2L \tag{2.28a}$$

or

$$\int_{G \cap C} dG = L. \tag{2.28b}$$

In other words the measure of the set of straight lines that intersect a convex curve is its length.

Suppose a curve C of length L is contained in a convex curve C_1 of length L_1. Consider all straight lines that intersect C_1. The expected number of intersections of such lines with C is given by $\bar{n}$,

$$\bar{n} = \frac{\int n(\rho, \theta)\, dG}{\int dG} = \frac{2L}{L_1}. \tag{2.29}$$

We may remark that $dG/\int dG$ is the probability that a line (p, θ) belongs to a small piece dG and $n(p, \theta)$ is its number of intersections with C. This shows that if a curve of length L can be contained inside a curve of length L_1, there exist straight lines which cut it in $2L/L_1$ points at least.

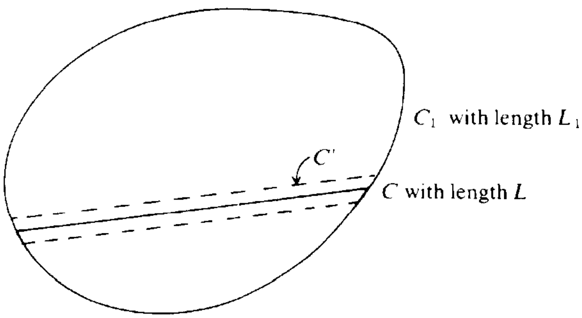


FIG. 2.3

Suppose again C_1 is a convex curve of length L_1 and let C be a chord of C_1 of length L. Construct a rectangle C' to approximate C with approximate perimeter $2L$. (See Fig. 2.3.) Then the expected number of intersections with C' and hence with C, of lines that intersect C_1 is given by

$$\frac{\int_{G \cap C'} dG}{\int_{G \cap C} dG} = \frac{2L}{L_1} \quad \text{since } C', C_1 \text{ are convex.} \tag{2.30}$$

Therefore we can calculate the probability of two lines intersecting within C_1 and this is given by

$$\int \frac{2L(p, \theta)}{L_1} \cdot \frac{dp\, d\theta}{\int dp\, d\theta}; \tag{2.31}$$

that is $2L(p, \theta)/L_1$ is the probability of a line intersecting a fixed line (p, θ) with

length $L(p, \theta)$ and $dp\, d\theta/(\int dp\, d\theta)$ is the probability of line $(p, \theta)\ \varepsilon$ a small piece dG. Thus the probability that the two lines intersect is

$$\frac{2}{L_1^2}\int L(p,\theta)\,dp\,d\theta = \frac{2}{L_1^2}\int_0^{\pi} d\theta \int L(p)\,dp = \frac{2\pi A}{L_1^2} \tag{2.32}$$

where A is the area of C_1, $\int L(p)\,dp = A$ for fixed θ. Finally the expected length of a chord intersecting C_1 is given by

$$\frac{\int_{G\cap C_1} L(p,\theta)\,dG}{\int_{G\cap C_1} dG} = \frac{\pi A}{L_1}. \tag{2.33}$$

This is one of the ingenious results due to Crofton that we have mentioned previously in discussing the estimation of the ratio of the volume to the surface area of a convex body.

We now comment briefly on measure and density for straight lines in three dimensions. Represent a line in 3 dimensions by four parameters a, b, p, q, that is,

$$x = az + p, \qquad y = bz + q \tag{2.34}$$

and the intersection of two planes determines a line or

$$y = b\left(\frac{x-p}{a}\right) + q = \frac{bx}{a} - \frac{bp}{a} + q. \tag{2.35}$$

This representation omits the set of all lines parallel to the plane $z = 0$ but these lines will have measure zero and hence offer no difficulty. Now set

$$m(A) = \int F(a, b, p, q)\,da\,db\,dp\,dq \tag{2.36}$$

and we want $m(A)$ as usual invariant under translations and rotations. In this case we find in Kendall and Moran (1963, pp. 18–19) that

$$F(a, b, p, q) = (1 + a^2 + b^2)^{-2} \tag{2.37}$$

and hence the density for straight lines in three dimensions is

$$\frac{da\,db\,dp\,dq}{(1 + a^2 + b^2)^2} \tag{2.38}$$

and is unique up to a constant multiple.

Pairs of points in the plane. Now we discuss density and measure for pairs of points in the plane. Given a pair of points P_1, P_2 we can determine them uniquely by four coordinates (X_1, Y_1, X_2, Y_2). Then it can be shown that the invariant measure over some region A is

$$m(A) = \int_A dP_1\,dP_2 = \int_A dx_1\,dx_2\,dy_1\,dy_2. \tag{2.39}$$

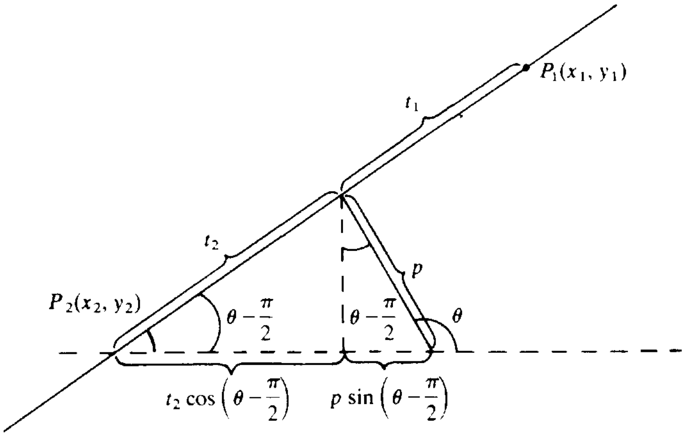


FIG. 2.4

Consider Fig. 2.4 where p, θ are the parameters for the line through the points P_1, P_2.

We wish to write $dP_1\, dP_2$ in terms of p, θ, t_1, t_2. Care should be taken in the interpretation of the directions of t_1, t_2. Now,

(2.40) $$[dP_1\, dP_2] = [dx_1\, dx_2\, dy_1\, dy_2] = f(p, \theta, t_1, t_2)[dp\, d\theta\, dt_1\, dt_2]$$

and

(2.41) $$x_i = p \cos\theta - t_i \sin\theta, \qquad y_i = p \sin\theta + t_i \cos\theta \quad \text{for } i = 1, 2.$$

Figure 2.4 verifies that

(2.42) $$x_2 = t_2\left[\cos\left(\theta - \frac{\pi}{2}\right)\right] + p \sin\left(\theta - \frac{\pi}{2}\right) = -t_2 \sin\theta + p \cos\theta.$$

Thus

(2.43) $$\begin{aligned} dx_i &= \cos\theta\, dp - (p \sin\theta + t_i \cos\theta)\, d\theta - \sin\theta\, dt_i, \\ dy_i &= \sin\theta\, dp + (p \cos\theta - t_i \sin\theta)\, d\theta + \cos\theta\, dt_i \end{aligned}$$

and

(2.44) $$[dx_i\, dy_i] = p[dp\, d\theta] + [dp\, dt_i] - t[d\theta\, dt_i]$$

and

(2.45) $$[dP_1\, dP_2] = [dx_1\, dy_1\, dx_2\, dy_2] = |t_2 - t_1|[dp\, d\theta\, dt_1\, dt_2]$$

via exterior multiplication. Thus

(2.46) $$[dP_1\, dP_2] = |t_2 - t_1|[dG\, dt_1\, dt_2]$$

and the measure for pairs of points is a function of the distance between them in terms of the new coordinates.

Now let K be a convex curve of length L and regional area A. Denote by $\sigma(p, \theta)$ the length of the chords determined by the straight line of (p, θ) on K. (See Figs. 2.5 and 2.6.) Define $I_n = \int \sigma^n \, dG$ where n is a positive integer and the integral is taken over all straight lines G which cut K. From before we have

$$(2.47) \qquad I_0 = \int dG = L, \qquad I_1 = \int \sigma \, dG = \int_0^{\pi} \int \sigma \, dp \, d\theta = \pi A.$$

If $r(P_1, P_2)$ is the distance between P_1, P_2 for any pair of points $P_1, P_2 \in K$ then define

$$(2.48) \qquad J_n = \int r^n \, dP_1 \, dP_2.$$

Thus we write

$$(2.49) \qquad J_n = \int |t_2 - t_1|^{n+1} \, dG \, dt_1 \, dt_2.$$

Fixing p, we see that θ determines a chord of length say $\sigma = b - a$ as in Fig. 2.6. Fix t_1 and integrate t_2 over $[a, b]$ and we have

$$(2.50) \qquad \begin{aligned} J_n &= \int dG \, dt_1 \left[\int_{t_1}^{b} (t_2 - t_1)^{n+1} \, dt_2 + \int_a^{t_1} (t_1 - t_2)^{n+1} \, dt_2 \right] \\ &= \frac{1}{n+2} \int dG \int_a^b ((b - t_1)^{n+2} + (t_1 - a)^{n+2}) \, dt_1 \\ &= \frac{2}{(n+2)(n+3)} \int (b-a)^{n+3} \, dG = \frac{2}{(n+2)(n+3)} \int \sigma^{n+3} \, dG \\ &= \frac{2}{(n+2)(n+3)} I_{n+3}. \end{aligned}$$

This produces the relations

$$(2.51) \qquad \begin{aligned} J_n &= \frac{2}{(n+2)(n+3)} I_{n+3}, \qquad && n \geqq -1, \\ I_n &= \frac{n(n-1)}{2} J_{n-3}, && n \geqq 2. \end{aligned}$$

Then we have the following

$$(2.52) \qquad \begin{aligned} I_2 &= \int \sigma^2 \, dG = \int \frac{dP_1 \, dP_2}{r} = J_{-1}, \\ I_3 &= \int \sigma^3 \, dG = 3J_0 = 3 \int dP_1 \int dP_2 = 3A^2. \end{aligned}$$

This last result valid for any convex curve is another remarkable result due to Crofton.

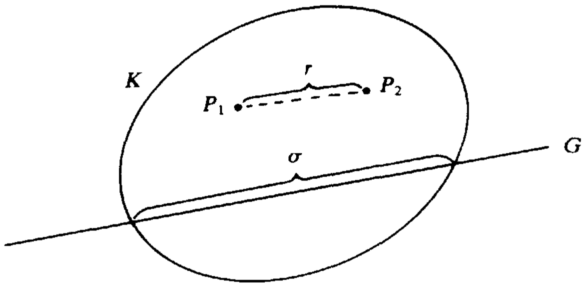


FIG. 2.5

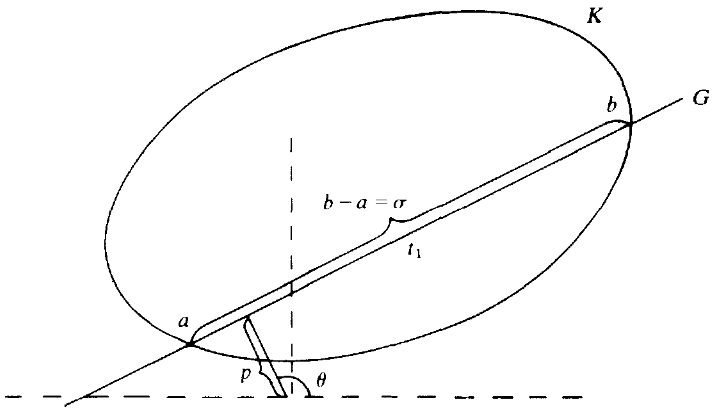


FIG. 2.6

Let $\bar{r}$ be the mean distance between a pair of points belonging to K. Then

(2.53a)
$$\bar{r}=\frac{\int r\,dP_1\,dP_2}{\int dP_1\,dP_2}=\frac{J_1}{A^2}=\frac{I_4}{6A^2}$$

or

(2.53b)
$$I_4=6\bar{r}A^2.$$

There are relationships between the I_n and J_n. For example

(2.54a)
$$I_0^2-4I_1\geqq 0$$

or

(2.54b)
$$L^2\geqq 4\pi A,$$

the classical "isoperimetric inequality". The equality holds only when K is a circle. Other inequalities due to Blaschke and Carleman respectively are

(2.55)
$$(15)^2\pi^8I_4^2-2^{16}I_1^5\geqq 0,\qquad (16)^2I_1^3-3^2\pi^4I_2^2\geqq 0.$$

In both cases equality holds only for the circle.

Suppose we consider circles only and let K be a circle of radius R. Then

$$I_0 = L = 2\pi R,$$

(2.56)
$$I_1 = \pi A = \pi^2 R^2,$$
$$I_2 = \frac{16^2 I_1^3}{3^2 \pi^4} = \frac{16}{3}\frac{\pi^3 R^2}{\pi^2} = \frac{16\pi}{3} R^3,$$
$$I_4 = \frac{2^{16} I_1^3}{15^2 \pi^8} = \frac{2^8}{15}\frac{\pi^5 R^5}{\pi^4} = \frac{256\pi R^5}{15}.$$

Hence for any pair of points belonging to the circle,

(2.57)
$$E\left(\frac{1}{r}\right) = \frac{\int (1/r)\, dP_1\, dP_2}{\int dP_1\, dP_2} = \frac{I_2}{A^2} = \frac{16\pi R^3}{3\pi^2 r^4} = \frac{16}{3\pi R},$$

(2.58)
$$E(r) = \frac{\int r\, dP_1\, dP_2}{\int dP_1\, dP_2} = \frac{I_4}{6A^2} = \frac{256\pi R^5}{6 \cdot 15\pi^2 R^4} = \frac{128R}{45\pi}.$$

The last two expectations arise in statistics from time to time and have been developed in different ways.

We now offer a brief presentation of measure and density for planes in three dimensions (Santaló (1953)). A plane is specified by three parameters, ρ, θ, φ (the usual spherical coordinates), where ρ is the distance from the origin to the plane, θ is the angle the projection of ρ on the xy-plane makes with the x-axis, φ is the angle between ρ and the z-axis. The ranges are

(2.59)
$$0 \leqq \rho < \infty, \quad 0 \leqq \theta \leqq 2\pi, \quad 0 \leqq \varphi \leqq \pi.$$

The equation of a plane with this representation is given by

(2.60)
$$x \sin\theta \cos\varphi + y \sin\theta \sin\varphi + z \cos\theta = \rho.$$

If we again seek an invariant measure under translations and rotations we will obtain the density

(2.61)
$$f(\rho, \theta, \varphi)\, d\theta\, d\rho\, d\varphi = \sin\theta\, d\theta\, d\rho\, d\varphi$$

and if A is a subset of the parameter space,

(2.62)
$$m(A) = \int_A \sin\theta\, d\theta\, d\rho\, d\varphi$$

is an invariant measure for planes in three dimensions. If the equation of the plane is written as

(2.63)
$$ux + vy + wz = 1,$$
$$-\infty < u < \infty, \quad -\infty < v < \infty, \quad -\infty < w < \infty$$

we can express the density in terms of u, v, w since

(2.64)
$$u = -\frac{\sin\theta\cos\varphi}{\rho}, \quad v = -\frac{\sin\theta\sin\varphi}{\rho}, \quad w = -\frac{\cos\theta}{\rho}.$$

By exterior multiplication we obtain

$$\sin\theta[d\theta\, d\rho\, d\varphi] = \frac{1}{(u^2+v^2+w^2)}[du\, dv\, dw]. \tag{2.65}$$

This representation will be helpful when we look at the Sylvester problem in three dimensions.

This page intentionally left blank

CHAPTER 3

Random Lines in the Plane and Applications

In a previous chapter, we provided a proof by Morton (1966) that for a set of rectifiable curves contained in a plane and separate set of rectifiable curves constrained to be at random on a particular region of the plane, the expected number of intersections between the two sets of curves is given by

$$E = \frac{2L_C L_D}{\pi A} \tag{3.1}$$

where L_C is the total length of the rectifiable curves C_i, $i = 1, 2, \cdots, n_i$, contained in the plane, L_D is the total length of the rectifiable curves D_j, $j = 1, 2, \cdots, n_j$, placed at random in the plane, A is the area of the region of the plane which is as defined previously; that is, it is the set of points of distance h from any arbitrary point of D, $r_I \leqq h \leqq r_G$, where r_G is the radius of the circle centered so that it circumscribes C and r_I is the shortest distance from the center to C. This condition dictates intersections. Also the angles, θ, between the two sets at their points of intersection are distributed as $\frac{1}{2}\sin\theta$.

This is a fascinating statement and now we examine the number of intersections of random lines in the plane and their angles of intersection. There are some earlier results for what we term the isotropic case, to be defined soon, by Miles (1964) who derived the expected number of intersections and by Wolfwitz (1949) who we previously mentioned derived the distribution of the intersecton angles.

Questions of the variance, higher moments, and the distribution of the number of intersections are of much interest. Directly related to this are the distributions of the magnitude of areas of polygons and number of sides of polygons formed by random lines in the plane. We now turn to this topic.

An initial contributor to this subject is Goudsmit (1945) who was motivated by some problems in physics—originally the positioning of tracks in cloud-chamber experiments and subsequently some other topics. To avoid difficulties with the notion of infinite lines in the plane, Goudsmit considered the analogous problem on the sphere. In that case the straight lines are replaced by great circles on the sphere. These great circles will intersect an arbitrarily chosen equator and because of symmetry we need only consider a hemisphere. Then we can study the distribution of the areas of the regions on the surface of a hemisphere that are formed by a large number of "great circles", i.e., circles formed by the intersections with planes through the center of the sphere. Each of these circles can be defined by the coordinates of one of its poles on the surface of the sphere

and we can then assume that each of these poles is uniformly distributed over the surface of the sphere. This provides the random mechanism by which N great circles emerge. By letting N go to infinity we obtain the analogue to an infinite number of line segments in the plane. Then by letting the radius of the sphere increase without bound we obtain the lines in the plane.

First Goudsmit presented a simplified version of the problem which can be solved completely without resorting to the sphere. Assume a plane is then cut into rectangular fragments. The lines in each set are distributed at random in the following way. Let the density of the lines be such that the x-axis is intersected by one line per unit length *on the average* and similarly for the y-axis. That is, the number of points of intersection is given by a Poisson random variable with parameter $\lambda = 1$. The problem is now reduced to that of two independent distributions for a number of random points on a line, one for the x-axis and one for the y-axis. Consider one rectangular fragment. The probability that its horizontal dimension is between η and $\eta + d\eta$ and its vertical dimension between ξ and $\xi + d\xi$ is given by

$$f(\xi, \eta)\, d\xi\, d\eta = \exp\,[-(\xi + \eta)]\, d\xi\, d\eta \tag{3.2}$$

since each length is given by the exponential distribution.

We are, however, only interested in the area $\sigma = \xi\eta$. Let

$$\sigma = \xi\eta, \qquad \mu = \xi + \eta$$

$$f(\xi, \eta)\, d\xi\, d\eta = \exp\,[-\mu]\, d\mu\, d\sigma/(\mu^2 - 4\sigma)^{1/2} \tag{3.3}$$

$$\text{because} \quad d\xi\, d\eta = \frac{d\mu\, d\sigma}{(\mu^2 - 4\sigma)^{1/2}}.$$

Note that $\mu^2 \geqq 4\sigma$. Thus the probability that the area of a rectangular fragment is σ is given by

$$F(\sigma)\, d\sigma = d\sigma \int_{\sqrt{4\sigma}}^{\infty} \frac{e^{-\mu}\, du}{(\mu^2 - 4\sigma)^{1/2}}. \tag{3.4}$$

The expectations of powers of σ are easily obtained.

$$E(\sigma^k) = \int_0^\infty \int_0^\infty \xi^k \eta^k e^{-(\xi+\eta)}\, d\xi\, d\eta = (k!)^2. \tag{3.5}$$

Thus $E(\sigma) = 1$. This value arises because the assumption of one line per unit length on the average has standardized these expectations in this way.

Now we return to the general problem and consider the distribution on a sphere. Assume that there are N halves of great circles distributed at random on a half sphere, not counting the "equator". (See Fig. 3.1.)

We can derive the following properties. The N lines divide the half sphere into $[\frac{1}{2}N(N+1)+1]$ fragments; and on the average each fragment has four sides when N increases indefinitely. The first statement can be proved by induction—the Nth line intersects all $(N-1)$ previous lines and in doing so cuts each of N

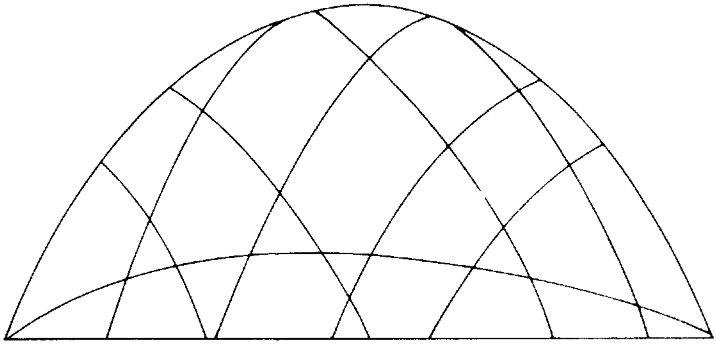


FIG. 3.1

fragments into two, thus adding just N fragments to the total. We started with one fragment, the whole half sphere; thus the number of fragments is given by

$$1+\sum_{i=1}^{N} i=\tfrac{1}{2}N(N+1)+1. \tag{3.6}$$

Now for the second statement. As we have already said, each of the N great circle halves is cut by the others into N segments. Each one of these segments serves as side to two adjacent fragments. In addition the "equator" is cut up into $2N$ segments bordering one fragment each. The total number of sides to the fragments is therefore

$$2N^2+2N. \tag{3.7}$$

The average number of sides per fragment is

$$\frac{2N^2+2N}{\frac{1}{2}(N^2+N)+1} \tag{3.8}$$

and this approaches 4 as $N\to\infty$.

The total surface area of the half sphere is $2\pi R^2$ or 2π since $R=1$, and since there are $\frac{1}{2}(N^2+N)+1$ fragments, the average area of a fragment is

$$\frac{2\pi}{\frac{1}{2}(N^2+N)+1} \tag{3.9}$$

or asymptotically it is

$$\frac{4\pi}{N^2}. \tag{3.10}$$

The total perimeter of the $\frac{1}{2}(N^2+N)+1$ fragments is twice the total length of all the great circles in the half-sphere, i.e., $2\pi N$, plus the circumference of the equator; thus the average perimeter of the fragment is

$$\frac{2\pi N+2\pi}{\frac{1}{2}(N^2+N)+1} \tag{3.11}$$

or asymptotically it is

$$\frac{4\pi}{N} \tag{3.12}$$

and the average length of each side is asymptotically

$$\frac{4\pi}{N}\left(\frac{1}{4}\right)=\frac{\pi}{N}. \tag{3.13}$$

We can now suppose that N becomes very large and consider the distribution of regions in a small circle of radius r on the surface of the sphere. The number of great circles which intersect this small circle will be asymptotically equal to rN (recall $0<r<1$) i.e., essentially the proportion of the small radius r to the unit radius multiplied by N the number of great circles. Proceeding to the limit and considering the region inside this small circle as approximately a plane circle, we find that the average area of the regions in a plane formed by random lines requires rescaling. This is developed now for Goudsmit's development. In the following sections on Poisson fields we will return to this point again.

Let us examine this in more detail going over to the plane case from the circle on the sphere and letting N and the radius of the sphere become infinite. We must watch the rescaling or standardization quite carefully. Suppose we wish to adjust the rescaling so that the average area of the fragments is unity; then the area of the half sphere has to be $N^2/2$ for large N (recall that there are $[\frac{1}{2}(N^2+N)+1]$ fragments, therefore

$$\frac{N^2/2}{\frac{1}{2}(N^2+N)+1}\to 1.$$

The radius of the half sphere is then $\frac{1}{2}N/\sqrt{\pi}$, (i.e., $2\pi(\frac{1}{2}N/\sqrt{\pi})^2=N^2/2$) and the length of the great circle half is $(\frac{1}{2}N/\sqrt{\pi})\pi=\frac{1}{2}N\sqrt{\pi}$. The mean length of the segments into which the great circles are cut is thus $\sqrt{\pi}/2$ and not unity as it was in the simplified rectangular case.

New let us look at the mean square area of the fragments in the general case. Consider two arbitrarily chosen points and ask for the probability that they happen to be in the same fragment. *This probability can be expressed in terms of the mean square area of the fragments.* Next consider the line which is determined by the two points and ask for the probability that the two points fall both in one of the segments into which the line is divided by all the other lines. This latter probability can be computed and thus the mean square area obtained.

The probability that the first arbitrarily chosen point lies in a fragment of size between σ and $\sigma+d\sigma$ is given by the fraction of the total area which is covered by such fragments, namely

$$\frac{\sigma SG(\sigma)\,d\sigma}{\int \sigma SG(\sigma)\,d\sigma} \tag{3.14}$$

where $G(\sigma)\,d\sigma$ is the density for area, S is the number of fragments in the total area.

The probability P_2 that the two points lie in the same fragment irrespective of its size is equal to the product of these two expressions integrated over all sizes

(3.15) $$P_2 = \frac{\int \sigma^2 SG(\sigma)\, d\sigma}{[\int \sigma SG(\sigma)\, d\sigma]^2} = \frac{E\sigma^2}{S[E\sigma]^2}.$$

We next consider the probability that the second point lies at a distance l to $l + dl$ from the first. This is given by the area of a ring divided by the total area (see Fig. 3.2)

(3.16) $$2\pi l\, dl/\text{total area}.$$

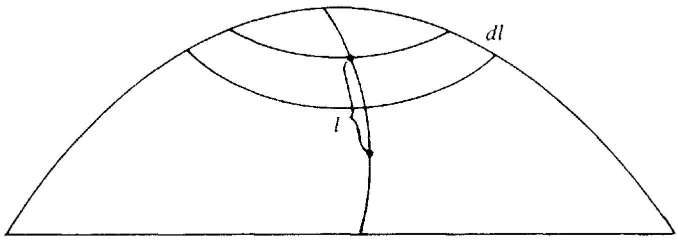


FIG. 3.2

Through the two points we pass a line which will be cut into segments by all the lines already present. The chance that no intersection will occur between the two chosen points is $\exp[-l/(\frac{1}{2}\sqrt{\pi})]$. The factor $\sqrt{\pi}/2$ arises from the rescaling discussed above; the mean distance between intersections is $\frac{1}{2}\sqrt{\pi}$.

The probability that regardless of their distance the two points are not separated by one of the lines is thus given by

(3.17) $$P_2 = \int 2\pi l\, dl\, e^{-l/((1/2)\sqrt{\pi})}/(SE(\sigma)), \qquad P_2 = \frac{1}{2}\pi^2/(S \cdot E(\sigma)).$$

Recall that $S \cdot E(\sigma) = $ total area, but by the rescaling we have made $E(\sigma) = 1$, and so

(3.18) $$P_2 = \frac{1}{2}\pi^2/S.$$

Now compare with $P_2 = E(\sigma)^2/(S \cdot [E(\sigma)]^2)$ and we get

(3.19) $$\frac{E(\sigma)^2}{[E(\sigma)]^2} = \frac{1}{2}\pi^2.$$

Note that this differs from the rectangular fragments case which gave the value 4 for this ratio.

To this point, Goudsmit is attempting to provide a random mechanism to produce lines in the plane and to develop some of the manifestations of this randomness such as the ratio of the second moment to the square of the first

moment. This knowledge can provide some check on the tenability of a randomness conjecture for the positioning of tracks in cloud-chamber experiments. Obviously knowledge of higher moments would provide more information and shortly we will present a derivation of the third moment employing Goudsmit's method of obtaining the second moment. The fourth moment and higher moments are not yet known.

First, however, we characterize a class of random lines in the plane that is unaffected by choice of origin or coordinate axes. It will turn out that Goudsmit's randomness mechanism is essentially leading to this class of random lines which is termed the Poisson field of random lines in the plane. Another name we shall employ is isotropic lines to distinguish it from some extensions when subsequently we view highway traffic flow models from some anisotropic line models.

Poisson field for random lines in the plane. Any line in the (x, y) plane can be represented as

$$p = x\cos\theta + y\sin\theta \qquad (-\infty < p < \infty, 0 \leqq \theta < \pi) \tag{3.20}$$

where p is the signed length of the perpendicular to the line from an arbitrary origin, O, and θ is the angle this perpendicular makes with the x-axis. Note that if the intersection of the perpendicular with the line is in the third or fourth quadrant, p is taken to be negative. A set of lines $\{(p_i, \theta_i): i = 0, \pm 1, \pm 2, \cdots\}$ are a Poisson field under the following conditions.

1. The distances $\cdots \leqq p_{-2} \leqq p_{-1} \leqq p_0 \leqq p_1 \leqq p_2 \leqq \cdots$ of the lines from an arbitrary origin, O, constitute the coordinates of the events of a Poisson process with constant intensity, say τ. Thus, the number of p_i in an interval of length L has a Poisson distribution with mean τL.

2. The orientations θ_i of each line with a fixed but arbitrary axis in the plane are independent and have a uniform distribution in the interval $(0, \pi)$.

This is a reasonable representation of random lines in the plane and we label it the Poisson field. Note that for a fixed line segment, the intersections are uniformly distributed over a range equal to the length of the line segment and thus have density proportional to dp leading to a joint density proportional to $dp\,d\theta$. This definition of randomness for lines in the plane is natural because the randomness is unaffected by the choice of origin or x-axis since $dp\,d\theta$ is the invariant density under rotations and translations.

One basic feature of randomness of lines generated by a Poisson field is the following result. The number of random lines intersecting any arbitrary line segment of length x is Poisson distributed with mean $2\tau x/\pi$. To prove this result we note that the randomness of the lines in the plane is not affected by rigid motion transformations and thus we can choose the origin and the axis orientation so that the line segment corresponds to the horizontal axis from $-x/2$ to $x/2$. Let N_x denote the number of lines intersecting the x-axis from $-x/2$ to $x/2$. Then we wish to show

$$P[N_x = n] = e^{-2\tau x/\pi}(2\tau x/\pi)^n/n!. \tag{3.21}$$

Let N_p denote the number of random lines whose signed distance, p, to the origin is between $-x/2$ and $x/2$. Then

$$P[N_x = n] = \sum_{m=0}^{\infty} P[N_x = n \mid N_p = m] P[N_p = m]. \tag{3.22}$$

Clearly no line can intersect the x-axis between $-x/2$ and $x/2$ unless its minimum distance to the origin is between $-x/2$ and $x/2$. Thus N_p must exceed N_x; that is,

$$P[N_x = n \mid N_p = m] = 0 \quad \text{for } n > m \tag{3.23}$$

and therefore

$$P[N_x = n] = \sum_{m=n}^{\infty} P[N_x = n \mid N_p = m] P[N_p = m]. \tag{3.24}$$

Let $\mu = P[N_x = 1 \mid N_p = 1]$. Then since the random lines are independent

$$P[N_x = n \mid N_p = m] = \binom{m}{n} \mu^n (1-\mu)^{m-n} \quad \text{for } m \geqq n. \tag{3.25}$$

By the Poisson field notion with intensity τ for random lines in the plane, we may write

$$P[N_p = m] = e^{-\tau x} (\tau x)^m / m!. \tag{3.26}$$

Thus

$$\begin{aligned} P[N_x = n] &= \sum_{m=n}^{\infty} \frac{m!}{n!(m-n)!} \mu^n (1-\mu)^{m-n} \frac{e^{-\tau x}(\tau x)^m}{m!} \\ &= \frac{e^{-\tau x}}{n!} \mu^n (\tau x)^n \sum_{k=0}^{\infty} \frac{(1-\mu)^k (\tau x)^k}{k!} \quad \text{where } k = m-n \\ &= \frac{(\mu \tau x)^n}{n!} e^{-\mu \tau x} \end{aligned} \tag{3.27}$$

since

$$\sum_{k=0}^{\infty} \frac{(1-\mu)^k (\tau x)^k}{k!} = \sum_{k=0}^{\infty} \frac{[(1-\mu)\tau x]^k}{k!} = e^{\tau x - \mu \tau x}. \tag{3.28}$$

Now we evaluate μ, the probability that a line whose minimum (signed) distance to the origin is between $-x/2$ and $x/2$ will intersect the x-axis between $-x/2$ and $x/2$. Let η denote the x-intercept of the line in Fig. 3.3. Then $\eta = p \sec \theta$ and

$$\begin{aligned} \mu &= P[|\eta| \leqq x/2 \mid |p| \leqq x/2] \\ &= P[-x/2 \leqq p \sec \theta \leqq x/2 \mid |p| \leqq x/2] \\ &= 2 \int_0^{x/2} P[-x/(2p) \leqq \sec \theta \leqq x/(2p) | p] \, dp/x \end{aligned} \tag{3.29}$$

where the last expression takes into account the symmetry about the origin of the density dp/x of p given that $|p| \leqq x/2$. The density of θ is $d\theta/\pi$. Let $y = \sec\theta$. Then $dy = \tan\theta \sec\theta \, d\theta = y\sqrt{y^2-1}\, d\theta$ and the density of y is $dy/(y\sqrt{y^2-1})$. Thus

$$\begin{aligned} P[-x/(2p) \leqq \sec\theta \leqq x/(2p) \mid p] &= \frac{1}{\pi}\int_{-x/(2p)}^{x/(2p)} \frac{dy}{y\sqrt{y^2-1}} \\ &= \frac{2}{\pi}\cos^{-1}\left(\frac{2p}{x}\right), \qquad 0 \leqq p \leqq 1. \end{aligned} \tag{3.30}$$

Hence

$$\begin{aligned} \mu &= \frac{2}{\pi}\int_0^{x/2} \cos^{-1}\left(\frac{2p}{x}\right)\left(\frac{2}{x}\right) dp \\ &= 2/\pi \end{aligned} \tag{3.31}$$

and we have

$$P[N_x = n] = e^{-2\tau x/\pi}(2\tau x/\pi)^n/n!. \tag{3.32}$$

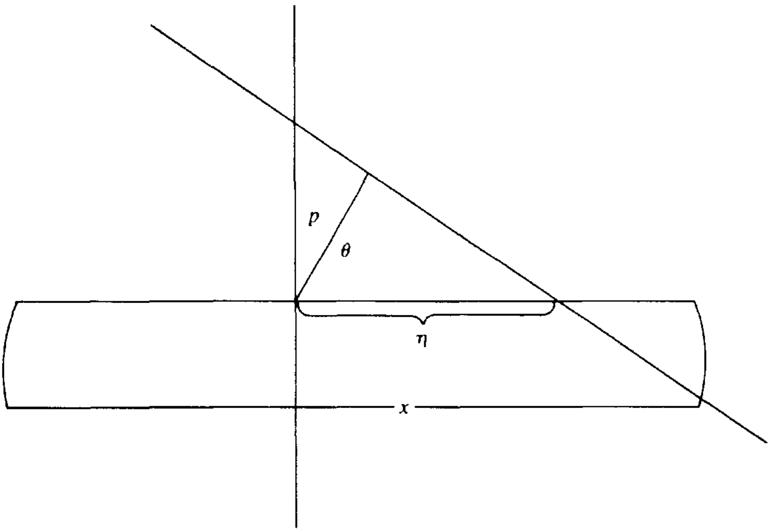


FIG. 3.3

The number of random lines intersecting disjoint sections of a line are clearly independent. Thus the points of intersection of the random lines with any arbitrary line constitute a Poisson process of intensity $2\tau/\pi$. This result without the development appears in Miles (1964).

Previously we derived some of the statistical properties of the polygons into which the random lines divide the plane. When the number of random lines is finite or the area of the plane under consideration is finite, difficulties arise with

edge effects and infinities. These problems can be avoided by considering a half sphere on whose surface are random great circle arcs whose poles are uniformly distributed over the surface of the half sphere. As the radius of the half sphere is increased toward infinity the random great circles become random straight lines in a Poisson field. By considering random great circles and the fragments into which they divide the surface of the half sphere and letting the radius of the sphere increase to infinity, we have obtained properties of random lines in the plane, namely the mean area, mean perimeter, and mean number of sides of the polygons into which the plane is divided.

As we let n increase toward infinity we must also increase the radius of the sphere in such a way that the limiting density of lines in the plane is τ. Since the points of intersection on any line in the Poisson field of lines in the plane constitute a Poisson process of density $2\tau/\pi$, the mean separation between intersections is $\pi/(2\tau)$. If R is the radius of the sphere, the mean separation between intersections on a great circle is $\pi R/n$. Thus we should choose R to be $\frac{1}{2}n/\tau$ so that the limiting density (as $n \to \infty$ and $R \to \infty$) of the lines in the plane is τ. Then the results we obtained before (from Goudsmit) can now be written in the following way: The surface area of the half sphere then becomes $2\pi R^2 = \frac{1}{2}\pi n^2/\tau^2$. Thus as $n \to \infty$ the fragments on the surface of the half sphere become polygons in the plane and the average area of the polygons is π/τ^2.

The average perimeter of a fragment can be obtained in a similar way and it is

$$\frac{\pi(n^2+n)/\tau}{\frac{1}{2}(n^2+n)+1} \tag{3.33}$$

which as $n \to \infty$ becomes $2\pi/\tau$. Consequently the average length of a side of a polygon, i.e., the average length of the segments into which the random lines are cut, is $\frac{1}{2}\pi/\tau$.

Henceforth we shall adopt the letters N, L, and A to denote the number of sides, the perimeter, and the area, respectively, of a polygon. Using the fact that as $n \to \infty$ the average values of N, L and A approach the mean values, we have obtained:

$$E[N]=4, \quad E[L]=2\pi/\tau, \quad E[A]=\pi/\tau^2. \tag{3.34}$$

If one wanted to test whether a given set of lines in the plane were, in fact, a Poisson field, it would be desirable to have second moments as well as these mean values. The value of $E[A^2]$ was first derived by Goudsmit. His approach consists of using two different methods to evaluate the probability that two random points r_1 and r_2 lie in the same polygon. One method results in an expression involving $E[A^2]$ and the other method results in an expression which can be explicitly evaluated. The first method is as follows:

Let $h(\sigma)$ be the probability density of the random variable A i.e., $\int_{\sigma_1}^{\sigma_2} h(\sigma)\, d\sigma$, $P[\sigma_1 < A < \sigma_2]$, the fraction of all the polygons which have areas between σ_1 and σ_2. For small $d\sigma$ the fraction of the number of polygons which have areas between σ and $\sigma + d\sigma$ is $h(\sigma)\, d\sigma$. Let M be the total number of polygons in

some large area D. Then for large M, $Mh(\sigma)\,d\sigma$ is approximately the number of polygons with areas between σ and $\sigma+d\sigma$, and $\sigma Mh(\sigma)\,d\sigma$ is approximately the total area of such polygons. Then the probability that r_1 lies in a polygon with area between σ and $\sigma+d\sigma$ is (asymptotically as $M\to\infty$ and $D\to\infty$) $\sigma Mh(\sigma)\,d\sigma/D$.

The probability of the second point, r_2, lying in the same polygon is σ/D. Thus the probability of both points lying in the same polygon irrespective of its size is the product of the above expressions integrated over all sizes:

$$M\int_0^\infty \sigma^2 h(\sigma)\,d\sigma/D^2 = ME(A^2)/D^2. \tag{3.35}$$

The second method of evaluating the same probability is as follows. The probability that the second point lies at a distance x to $x+dx$ from the first is the area of the ring of width dx divided by the total area, or

$$2\pi x\,dx/D. \tag{3.36}$$

The probability that no line will divide the two points is the probability of no intersection on the line segment joining the two points, namely $e^{-2\tau x/\pi}$. The probability that irrespective of their distance apart the two points are not separated by one of the random lines is

$$\int_0^\infty 2\pi x\, e^{-2\tau x/\pi}\,dx/D = \frac{1}{2}\pi^3/(\tau^2 D). \tag{3.37}$$

Hence, equating the two expressions for the probability that two random points lie in the same polygon, we get

$$E(A^2) = \frac{1}{2}\pi^3 D/(\tau^2 M).$$

As D and the number of lines approach infinity

$$D/M \to E(A) = \pi/\tau^2 \quad \text{and so} \quad E(A^2) = \frac{1}{2}\pi^4/\tau^4. \tag{3.38}$$

Higher order moments. A variation of this basic method can be used to obtain moments involving A^3. Consider three points, r_1, r_2, and r_3, which are randomly placed in some large domain D. Let the random variable P be the perimeter of the triangle formed by the three points; i.e.

$$P = \|r_1 - r_2\| + \|r_2 - r_3\| + \|r_3 - r_1\|. \tag{3.39}$$

We shall average an arbitrary function $g(p)$ over all triples r_1, r_2, and r_3 that lie within a common polygon. First we derive the probability density function $f(p)$ of p.

$$\begin{aligned} f(p)\,\Delta p &\cong \Pr[p < P \leqq p + \Delta p] \\ &= \int_0^{p/2} \Pr\,[p < P \leqq p + \Delta p \mid \|r_1 - r_2\| = x][2\pi x\,dx/D). \end{aligned} \tag{3.40}$$

(Here we use Pr for probability to avoid confusion with perimeter P.) Given that $\|r_1 - r_2\|$ is x, the locus of points r_3 such that P is exactly p is an ellipse with major and minor axes

$$a(p,x)=\frac{1}{2}(p-x), \qquad b(p,x)=\frac{1}{2}(p^2-2xp)^{1/2}. \tag{3.41}$$

Thus, given that $\|r_1 - r_2\|$ is x, the probability that r_3 (randomly placed on the area D) will result in a value of P between p and $p+\Delta p$ is the difference (divided by D) between the area of the ellipses with axes $a(p+\Delta p, x)$ and $b(p+\Delta p, x)$ and with axes $a(p,x)$ and $b(p,x)$; i.e.,

$$[\pi a(p+\Delta p, x)b(p+\Delta p, x)-\pi a(p,x)b(p,x)]/D. \tag{3.42}$$

Substituting this expression for $\Pr[p<P\leqq p+\Delta p \mid \|r_1-r_2\|=x]$ in our equation for $f(p)\,\Delta p$ we can write

$$\begin{aligned} f(p) &= 2(\pi^2/D^2)\left[\int_0^{(p+\Delta p)/2} a(p+\Delta p, x)b(p+\Delta p, x)x\,dx - \int_0^{p/2} a(p,x)b(p,x)x\,dx/\Delta p\right] \\ &\quad -2(\pi^2/D^2)\int_{p/2}^{(p+\Delta p)/2} a(p+\Delta p, x)b(p+\Delta p, x)x\,dx/\Delta p. \end{aligned} \tag{3.43}$$

As Δp approaches zero, the last integrand on the right hand side approaches zero. Thus we get

$$\begin{aligned} f(p) &= 2(\pi^2/D^2)\frac{d}{dp}\left[\int_0^{p/2} a(p,x)b(p,x)x\,dx\right] \\ &= 2(\pi^2/D^2)\frac{d}{dp}\left[\frac{p^4}{60}-\frac{p^4}{210}\right] \\ &= (2/21)\pi^2p^3/D^2. \end{aligned} \tag{3.44}$$

The probability that the three points lie in the same polygon is the probability that no random line intersects the triangle with vertices r_1, r_2, and r_3. We have shown that this probability is $e^{-\tau p/\pi}$. Thus the mean of $g(p)$ over all r_1, r_2, and r_3 in a common polygon is

$$\frac{1}{D^2}\frac{2\pi^2}{21}\int_0^\infty g(p)p^3\,e^{-\tau p/\pi}\,dp. \tag{3.45}$$

Proceeding in the same way as with two points we get

$$\frac{M}{D^3}E\left[\int\int\int g(p)\,d\sigma_1\,d\sigma_2\,d\sigma_3\right]=\frac{1}{D^2}\frac{2\pi^2}{21}\int_0^\infty g(p)p^3\,e^{-\tau p/\pi}\,dp \tag{3.46}$$

so that

$$
(3.47) \qquad \begin{aligned} E\left[\int\int\int g(p)\,d\sigma_1\,d\sigma_2\,d\sigma_3\right] &= \frac{D}{M}\frac{2\pi^2}{21}\int_0^\infty g(p)p^3\,e^{-\tau p/\pi}\,dp \\ &\to E(A)\frac{2\pi^2}{21}\int_0^\infty g(p)p^3\,e^{-\tau p/\pi}\,dp \\ &= \frac{2\pi^3}{21\tau^2}\int_0^\infty g(p)p^3\,e^{-\tau p/\pi}\,dp. \end{aligned}
$$

Choosing $g(p)=1$, we get $E[A^3]=\frac{4}{7}\pi^7/\tau^6$. Letting $g(p)=p$ we get

$$
(3.48) \qquad E\left[\int\int\int (\|r_1-r_2\|+\|r_2-r_3\|+\|r_3-r_1\|)\,d\sigma_1\,d\sigma_2\,d\sigma_3\right]=\frac{2^4\pi^8}{7\tau^7}.
$$

Since $\iiint \|r_i-r_j\|\,d\sigma_1\,d\sigma_2\,d\sigma_3=RA^3$ for $i\neq j$, we get $E[RA^3]=2^4\pi^8/(21\tau^7)$.

A third variation of this method yields some of the moments involving L. We randomly place two points r_1 and r_2 in a large domain of area D and average an arbitrary function $g(\|r_1-r_2\|)$ over all pairs (r_1, r_2) such that the points fall inside a common polygon and each point lies within a small distance, w, of its boundary.

We shall consider four disjoint sets of random lines:

S: separating lines, that pass between r_1 and r_2;

C: common lines, within a distance w of r_1 and r_2 but not between them;

E_1 and E_2: end lines, within w of one point but not within w of the other point and not between them.

The lines in S are exactly the lines intersecting the line segment of length x between r_1 and r_2. We have already shown that the number of lines in S is Poisson distributed with mean $M_S=2\tau x/\pi$.

The lines in the union of the four sets are exactly the lines that pass within a distance w of the line segment between r_1 and r_2. Thus they are exactly the lines intersecting the convex region shown in Fig. 3.4. Since the mean number of random lines intersecting a convex figure of perimeter p is $\tau p/\pi$ and the region in the diagram has perimeter $2x+2\pi w$, we have

$$
(3.49) \qquad M_S+M_C+2M_E=2\tau(x+\pi w)/\pi
$$

(M_E denotes M_{E_1} which equals M_{E_2} by symmetry).

Similarly the lines in the union of S, C and E, are exactly the lines that intersect the convex region shown in Fig. 3.5 which has perimeter $[\pi w+2w\sin^{-1}(w/x)+2(x^2-w^2)^{1/2}]$. Thus we have

$$
(3.50) \qquad M_S+M_C+M_E=\tau[\pi w+2w\sin^{-1}(w/x)+2(x^2-w^2)^{1/2}]/\pi.
$$

Expanding the $\sin^{-1}$ term in a series and solving the last three equations, we get

$$
(3.51) \qquad \begin{aligned} M_E &= \tau w-\tau w^2/(\pi x)+o(w^3), \\ M_C &= 2\tau w^2/(\pi x)+o(w^3). \end{aligned}
$$

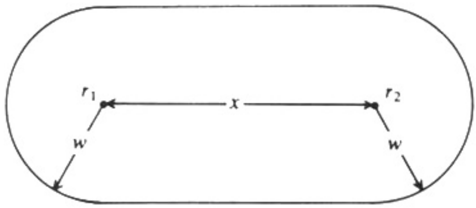


FIG. 3.4

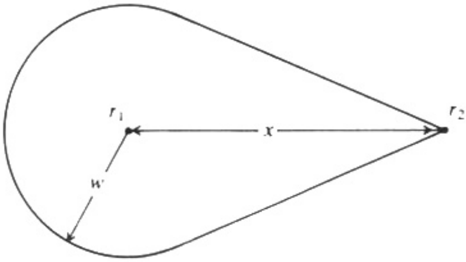


FIG. 3.5

Now we are able to calculate the probability that two points with separation x lie inside a common polygon, each point being at distance w or less from the boundary. This probability is the sum of the probabilities of the two mutually exclusive events:

$$(3.52)\qquad \begin{aligned} &W_1\colon\quad S=\Phi,\quad C=\Phi,\quad E_1\neq\Phi,\quad E_2\neq\Phi,\\ &W_2\colon\quad S=\Phi,\quad C\neq\Phi. \end{aligned}$$

Thus

$$(3.53)\qquad \begin{aligned} \Pr[W_1] &= \Pr[S=\Phi]\Pr[C=\Phi]\Pr[E_1\neq\Phi]\Pr[E_2\neq\Phi]\\ &= e^{-2\tau x/\pi}\, e^{-2\tau w^2/(\pi x)+o(w^3)}[1-e^{-\tau w+\tau w^2/(\pi x)+o(w^3)}]^2\\ &= \tau^2 w^2\, e^{-2\tau x/\pi}+O(w^3), \end{aligned}$$

$$(3.54)\qquad \begin{aligned} \Pr[W_2] &= \Pr[S=\Phi]\Pr[C\neq\Phi]\\ &= e^{-2\tau x/\pi}[1-e^{-2\tau w^2/(\pi x)+O(w^3)}]\\ &= \frac{2\tau}{\pi x} w^2\, e^{-2\tau x/\pi+O(w^3)} \end{aligned}$$

and

$$(3.55)\qquad \Pr[W_1\cup W_2]=\tau w^2[\tau+2/(\pi x)]\, e^{-2\tau x/\pi}+O(w^3).$$

This is the probability that r_1 and r_2 with separation x lie inside a common

polygon, each point less than w from the boundary. As before, the density of x is $2\pi x\,dx/D$. Thus the mean of $g(x)$ over all r_1 and r_2 in the same polygon and within w of its boundary is

$$(3.56)\qquad (2\pi\tau w^2/D)\int_0^\infty [\tau x+2/\pi]g(x)\,e^{-2\tau x/\pi}\,dx+O(w^3).$$

Let A_w denote the area of the strip of width w inside the boundary of the polygon. Then by the same arguments as in previous cases a second expression can be obtained for the mean of $g(\|r_1-r_2\|)$ over all r_1 and r_2 in the same polygon and near the boundary; namely,

$$(3.57)\qquad \frac{MA_w^2}{D^2}E\left[\int\int g(\|r_1-r_2\|)\frac{d\sigma_1\,d\sigma_2}{A_w^2}\right].$$

For small w, $d\sigma_i \cong w\,ds_i$ where ds denotes differential length along the boundary of the polygon. Equating our two expressions we get

$$(3.58)\qquad \frac{M}{D^2}w^2E\left[\int\int g(\|r_1-r_2\|)\,ds_1\,ds_2\right]$$
$$=(2\pi\tau w^2/D)\int_0^\infty [\tau x+2/\pi]g(x)\,e^{-2\tau x/\pi}\,dx+O(w^3).$$

Letting w go to 0 and M and D tend to infinity so that $D/M\to E(A)=\pi/\tau^2$, we get

$$(3.59)\qquad E\left[\int\int g(\|r_1-r_2\|)\,ds_1\,ds_2\right]=2\pi\int_0^\infty (2\tau+\pi x)g(x)\,e^{-2\tau x/\pi}\,dx.$$

Choosing $g(x)=1$ we arrive at the result

$$(3.60)\qquad E[L^2]=\pi^2(\pi^2+4)/(2\tau^4).$$

On the other hand, choosing $g(x)=x^2$ we get

$$(3.61)\qquad E\left[\int\int \|r_1-r_2\|^2\,ds_1\,ds_2\right]=\frac{\pi^4}{4\tau^4}[4+3\pi^2].$$

Let r_L be the center of gravity of the perimeter of the polygon. Then

$$(3.62)\qquad \|r_1-r_2\|^2=\|r_1-r_L\|^2+\|r_2-r_L\|^2+2\|r_1-r_L\|\,\|r_2-r_L\|\cos\alpha_{12}$$

where α_{12} is the angle at r_L between lines to r_1 and r_2. By symmetry

$$(3.63)\qquad \int\int \|r_1-r_L\|\,\|r_2-r_L\|\cos\alpha_{12}\,ds_1\,ds_2=0$$

and

$$(3.64)\qquad \int\int \|r_1-r_L\|^2\,ds_1\,ds_2=\int\int \|r_2-r_L\|^2\,ds_1\,ds_2.$$

Thus

$$2E\left[\int\int \|r_1 - r_L\|^2 \, ds_1 \, ds_2\right] = \frac{\pi^4}{4\tau^4}[4+3\pi^2] \tag{3.65}$$

i.e.,

$$E[LI_L] = \frac{\pi^4}{8\tau^4}[4+3\pi^2] \tag{3.66}$$

where I_L is the moment of inertia of the perimeter about its own center of gravity (i.e., $I_L = \int \|r - r_L\|^2 \, ds$ where r is on the perimeter).

Some additional features of random lines in the plane have been presented by R. E. Miles (1964) and are repeated here without proof.

1) For random lines obeying a Poisson field with intensity τ there are on the average τ^2/π intersections per unit area. The associated angles of intersection are mutually independent with common probability density $\frac{1}{2}\sin\Phi$ $(0 \leqq \Phi \leqq \pi)$. Thus random lines can be expected to intersect each other at angles close to right angles much more frequently than at angles close to 0° or 180°.

2) Let W be the diameter of the largest circle contained in a polygon formed by random lines as given above. Then

$$\Pr[W \leqq w] = 1 - e^{-\tau w},$$

or W is distributed as a negative exponential random variable with mean τ^{-1}.

3) Let L_k denote the perimeter of a k-sided polygon formed by the random lines. Then $2\tau L_k/\pi$ is χ^2 with $2(k-2)$ degrees of freedom. In particular, the perimeter of a triangle is negative exponential with mean value π/τ. A corollary is that the mean length of a side for a class of k-sided polygons is $(k-2)\pi/(k\tau)$ which increases to π/τ as $k \to \infty$. Thus the average length of a side of a many sided polygon can be expected to be about twice the average length of a side for the entire class of polygons and about three times the average length of a side of a triangle.

4) $\Pr[N=3] = 2 - \pi^2/6 \cong .3551$.

For $S \ll \tau^{-1}$ and $A \ll \tau^{-2}$ the probability densities of S and A are $(12-\pi^2)\tau/(6\pi) + O(\tau^2 S)$ and $c\tau A^{-1/2} + O(\tau^2)$ respectively where

$$c = \frac{1}{3\pi}\int_0^{\pi}\int_0^{\pi-\Phi} [2\sin\Phi \sin\psi \sin(\Phi+\psi)]^{1/2} \, d\psi \, d\Phi.$$

5) The random variables N, L, and A corresponding to the number of sides, perimeter, and area of a polygon have the covariance matrix

$$\begin{array}{c} \\ N \\ L \\ A \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad N \quad & \quad L \quad & \quad A \quad \end{array} \\
\begin{bmatrix}
(\pi^2-8)/2 & \pi(\pi^2-8)/(2\tau) & \pi(\pi^2-8)/(2\tau^2) \\
\pi(\pi^2-8)/(2\tau) & \pi^2(\pi^2-4)/(2\tau^2) & \pi^2(\pi^2-4)/(2\tau^3) \\
\pi(\pi^2-8)/(2\tau^2) & \pi^2(\pi^2-4)/(2\tau^3) & \pi^2(\pi^2-2)/(2\tau^4)
\end{bmatrix}
\end{array}.$$

Furthermore,

$$E[NA^2] = \pi^4(8\pi^2 - 21)/(21\tau^4),$$
$$E[LA^2] = 8\pi^7/(21\tau^5),$$
$$E[LA^{m-1}] = 2\tau E[A^m]/m \qquad (m = 1, 2, \cdots).$$

6) Let V be the set of intersections of the random lines. Define I_k as the set of line segments l such that l joins two points of V, there are exactly k other points of V on l, and l is part of one of the random lines. Similarly define J_k as the set of line segments l such that l joins two points of V, there are exactly k other points of V on l, and l is not a part of one of the random lines. Thus, for example, I_0 and J_0 are the sets of all sides and diagonals, respectively, of the polygons formed by the random lines in the plane. If S is the length of a member of I_k, then $4\tau S/\pi$ is χ^2 with $2(k+1)$ degrees of freedom. If T is the length of a member of J_k, then $4\tau T/\pi$ is χ^2 with $2(k+2)$ degrees of freedom. Miles presents similar results for random lines which have a nonzero thickness.

The derivation of the exact distribution of N, L and A is still an open question. Empirical distribution functions for these variables can be obtained by Monte Carlo methods. Some early results from a computer simulation of random lines in the plane by Stuart Dufour[1] are the following:

$$\begin{aligned} p_3 &= \Pr(N=3) \cong 0.36\\ p_4 &= \Pr(N=4) \cong 0.38\\ p_5 &= \Pr(N=5) \cong 0.19\\ p_6 &= \Pr(N=6) \cong 0.054\\ p_7 &= \Pr(N=7) \cong 0.010. \end{aligned} \tag{3.67}$$

For $\tau = 1$

$$\begin{array}{ll} \Pr(A \leqq 0.05) \cong 0.13 & \Pr(L \leqq 0.5) \cong 0.05\\ \Pr(A \leqq 0.10) \cong 0.18 & \Pr(L \leqq 1.0) \cong 0.11\\ \Pr(A \leqq 0.25) \cong 0.27 & \Pr(L \leqq 2.5) \cong 0.26\\ \Pr(A \leqq 0.50) \cong 0.38 & \Pr(L \leqq 5.0) \cong 0.51\\ \Pr(A \leqq 0.75) \cong 0.45 & \Pr(L \leqq 7.5) \cong 0.67\\ \Pr(A \leqq 1.00) \cong 0.50 & \Pr(L \leqq 10.0) \cong 0.79\\ \Pr(A \leqq 2.50) \cong 0.67 & \Pr(L \leqq 15.0) \cong 0.92\\ \Pr(A \leqq 5.00) \cong 0.80 & \Pr(L \leqq 20.0) \cong 0.98.\\ \Pr(A \leqq 10.00) \cong 0.90 & \\ \Pr(A \leqq 15.00) \cong 0.95 & \end{array} \tag{3.68}$$

[1] Unpublished memo.

These rough empirical distributions were obtained from an analysis of 947 polygons formed by 65 random lines. Only polygons within a radius of 32.5 were considered. Note the similarity between $\Pr(N=j)$ and $\Pr(Y=j-3)$ where Y is Poisson distributed with mean 1.

Along similar lines the following have been transmitted by R. E. Miles in a private communication to H. Solomon in September 1969.

(3.69)

p_3	.355065933 (exact)	p_8	.00207(0)
p_4	.3814(0)	p_9	.00026(0)
p_5	.189(5)	p_{10}	.000026(0)
p_6	.0586(5)	p_{11}	.0000021(5)
p_7	.0127(5)	p_{12}	.00000015(0).

Miles has some suspicions about these results for he conjectures that $p_n \sim e^{-n}$ as $n \to \infty$, an assumption not employed in obtaining the values listed above. Miles then fit the equation

$$p_n = e^{-n} + e^{-2n}(w + xn) + e^{-3n}(y + zn) \tag{3.70}$$

to the four known relations: $\sum p_n = 1$, $\sum np_n = 4$, $\sum n^2 p_n = (\pi^2/2) + 12$, $p_3 = 2 - (\pi^2/6)$ obtaining

p_3	.35507	p_8	.00237
p_4	.36686	p_9	.00047
p_5	.20304	p_{10}	.00010
p_6	.05530	p_{11}	.00003
p_7	.01195	p_{12}	.00001.

This seems to be rather inaccurate.

Miles also reports on two long runs of simulation of random polygons on a computer by a colleague wherein he lists the frequencies and relative frequencies as in Table 1.

To achieve three place accuracy, it appears that several hundred thousand polygons may be required!

Analytically, the problem seems too formidable. It is possible to express p_4 as a complicated 3-fold integral; also for the polygon containing the origin or any arbitrary fixed point in the plane, $p_3 = (\pi^2/6)[25 - 36 \log 2] = .07681381$.

Anisotropic lines in the plane: An application. It is possible to view the trajectory of a car produced by its time and space coordinates on the highway as a straight line in a plane if the car travels at a constant speed once it enters the highway and then never leaves the highway. We assume no change in a car's velocity when it overtakes another car, or is overtaken by another car. The Poisson process is the random device governing car entrance times or

TABLE 1

	I	II	Rel. frequencies (combined)
3	3635	14082	.35900
4	3952	15055	.38500
5	1893	7227	.18500
6	586	2163	.05600
7	141	503	.01300
8	29	98	.00260
9	3	5	.00016
10	0	0	.00000
11	0	0	.00000
12	0	1	.00002
	10,239	39,135	1.00078

equivalently car positions, and the speed distributions for each vehicle are assumed to be identically and independently distributed (i.i.d.). Thus we can view the trajectory of any car as a random line in the plane and we will shortly formalize this notion. The intersections of these random lines in the plane will represent time and space coordinates where an overtaking occurs.

Let one of the vehicles be an observer car (arbitrary line). The number of intersections of the arbitrary line (observer car) by the other lines determines the number of overtakings of slower cars made by the observer car plus the number of times it was overtaken by faster cars. We will develop this distribution and also the distribution of faster car overtakings of the observer car and overtaking of slower cars by the observer car. This requires a formulation for an anisotropic Poisson field of random lines from which a different proof of Rényi's result (1964) and the Weiss and Herman result (1962) can follow. These authors achieved their results without employing a geometrical probability context which we now develop.

We have already formalized the notion of straight lines distributed "at random" throughout the plane. We will describe the plane in terms of (t, x) coordinates, where subsequently the t axis will be employed to register time of arrival of cars at a fixed point on a highway and the x axis will in similar fashion report on spatial positions of cars on a highway at a fixed point in time. The time invariance property for Poisson processes will insure that the conditions will prevail at any point in time. Any line in the (t, x) plane can be represented as

$$p = t\cos\alpha + x\sin\alpha, \qquad -\infty < p < \infty, \quad 0 \leqq \alpha < \pi, \tag{3.71}$$

where p is the signed length of the perpendicular to the line from an arbitrary origin O, and α is the angle this perpendicular makes with the t axis. Note that if the intersection of the perpendicular with the line is in the third or fourth quadrant, p is taken to be negative. Assume the set of lines $\{(p_i, \alpha_i): i = 0, \pm 1, \pm 2, \cdots\}$ constitutes a Poisson field.

This definition of randomness for lines in the plane also has the property that the randomness is unaffected by the choice of origin or line to serve as t axis, since it can be demonstrated that except for a constant factor $\int dp\, d\alpha$ is the only invariant measure under the group of rotations and translations that transforms the line (p, α) to the line (p', α'). We now employ it as a point of departure to initiate discussion of a nonhomogeneous Poisson field of random lines. To achieve this we will relax one of the conditions and ask only that the α_i be identically and independently distributed (i.i.d.). This will permit a range of velocity distributions for the auto and make the model more realistic. The random lines are now labeled anisotropic lines in the plane.

For ease in the algebra of our traffic flow models, we will employ instead of α_i an angle formed by the intersection of the t axis with a line in the plane and we label this θ where $v = \tan\theta$. Also we will only be concerned with those lines where p_i falls in the second or fourth quadrant since this will yield all positive car velocities. The inclusion of the p_i in the first and third quadrants does not complicate the mathematical development, but they are not relevant. Thus, $\alpha = \frac{1}{2}\pi + \theta$ and the lines of interest will now be parametrized by (p, θ) where

$$p = -t \sin\theta + x \cos\theta, \qquad 0 \leqq \theta < \tfrac{1}{2}\pi. \tag{3.72}$$

Equation (3.72) takes care of the sign of p, for it insures that p will be positive if it is in the second quadrant and negative in the fourth quadrant.

The set $\mathscr{L}$ of lines $\{(p_i, \alpha_i): i = 0, \pm 1, \pm 2, \cdots\}$ is not anisotropic and we require invariant measure only under translation. We look into this situation because it will be helpful in our traffic flow models. Under this constraint, we now have the same conditions except that the orientation angles α_i of each line are i.i.d. random variables with common distribution function in the interval $(0, \pi)$. Thus, $\int dp\, d\alpha$ is no longer the appropriate measure and the origin can still be arbitrarily chosen but only at any point on a specific and fixed t axis because invariance is preserved now only under translation.

The orientations θ_i are independent and identically distributed with common distribution F in the interval $(0, \pi/2)$, and further the sequence of values $\langle\theta_i\rangle$ is independent of $\langle p_i\rangle$. This is equivalent to the statement that the velocities of cars, namely $v_i = \tan\theta_i$, are independent and identically distributed with common distribution G on $(0, \infty)$ and thus $\langle v_i = \tan\theta_i\rangle$ are independent of $\langle p_i\rangle$.

When $\theta_0 = 0$, $p_0 = 0$, the traffic flow is characterized by a distribution of time intercepts on the t axis; when $\theta_0 = \pi/2$, $p_0 = 0$, the traffic flow is characterized by a distribution of cars spaced along the x axis. For any other value of θ, the traffic flow is measured along a trajectory line. In the traffic literature, trajectories for low density traffic flow (no delays in overtaking) may be assumed to be linear in the time-space plane. Thus in any development, we must employ the appropriate measure to characterize distributions of traffic flow in such matters, for example, as distribution of number of overtakings. For our purposes where Poisson processes are the underpinning for traffic flow in both spatial and temporal processes the evaluation of the appropriate Poisson intensity parameter will be

paramount as will be the relationships between these parameters for different measures.

As noted above, each line is parametrized by (p, θ), where $0 \leqq \theta < \pi/2$, and p is the length of the perpendicular from the origin to the line. It is given a positive sign if the line cuts the t axis at time $t_0 > 0$, and a negative sign if it cuts the t axis at time $t_0 < 0$. A glance at Fig. 3.6 will indicate cars faster and slower than the observer car.

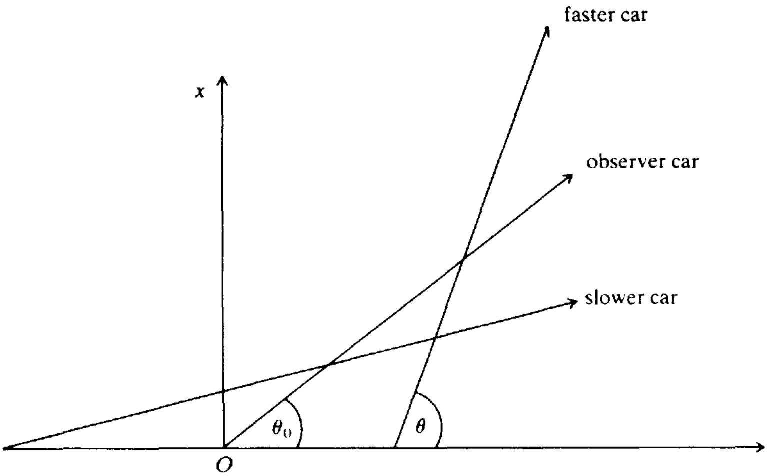


FIG. 3.6

Label N_L the number of cars intersecting the observer car before the point $t = L \cos \theta_0$, $x = L \sin \theta_0$, i.e., in the line segment L of the line $l = (0, \theta_0)$ stretching from the origin to the point. Let N_p denote the number of cars with the opportunity to intersect the observer car on this stretch. Then N_p is the number of cars with $-L \sin \theta_0 < p < L \cos \theta_0$.

Now any one of these N_L vehicles actually is overtaken or overtakes, given its θ, if

$$\text{(3.73)}\qquad \begin{aligned} &\text{(i)}\quad 0<\theta<\theta_0 \quad \text{and} \quad 0<-p<L\sin(\theta_0-\theta) \\ &\qquad\qquad\qquad\quad\ \ \text{or} \quad -L\sin(\theta_0-\theta)<p<0, \\ &\text{(ii)}\quad \theta_0<\theta<\pi/2 \quad \text{and} \quad 0<p<L\sin(\theta-\theta_0). \end{aligned}$$

Graphically we show (3.73) in Fig. 3.7.

Define $\mu = P\{N_L = 1 \mid N_p = 1\}$; then

(3.74)

$$\mu = P\{0 < \eta < L \mid \exists \text{ exactly one random line } (p, \theta) \text{ with } -L \sin \theta_0 < p < L \cos \theta_0\}$$

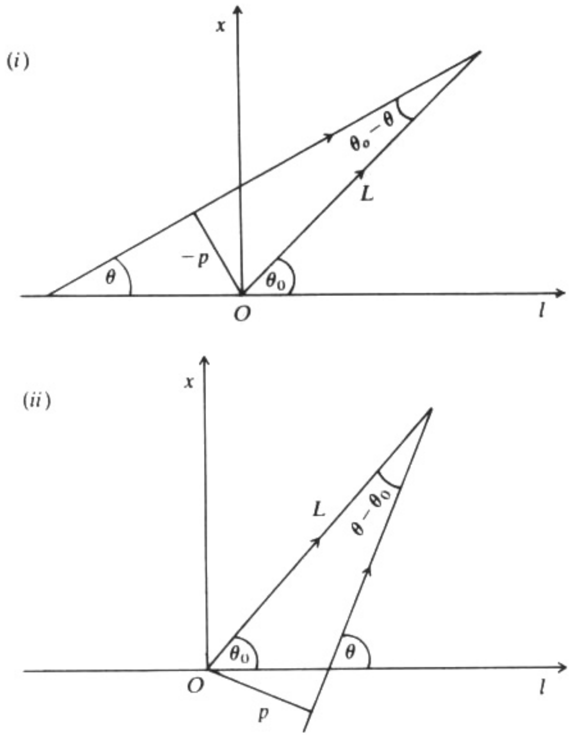


FIG. 3.7

and

(3.75) $$\mu=\left[\int_0^{\theta_0}\int_{-L\sin(\theta_0-\theta)}^{0}+\int_{\theta_0}^{\pi/2}\int_0^{L\sin(\theta-\theta_0)}\right]$$

$\cdot\, dF(\theta, p|\exists$ exactly one random line (p, θ) with $-L\sin\theta_0<p<L\cos\theta_0)$.

Now the distribution of θ is independent of the distribution of p and the condition depends only on p. Moreover the conditional distribution of p given $-L\sin\theta_0<p<L\cos\theta_0$ is uniform on the interval $(-L\sin\theta_0, L\cos\theta_0)$ since the p's are from a Poisson process.

Therefore we may write

(3.76a) $$\mu=\left[\int_0^{\theta_0}\int_{-L\sin(\theta_0-\theta)}^{0}+\int_{\theta_0}^{\pi/2}\int_0^{L\sin(\theta-\theta_0)}\right]\frac{dp}{L\cos\theta_0+L\sin\theta_0}\,dH(\theta)$$

or

(3.76b) $$\mu=\int_0^{\theta_0}\frac{L\sin(\theta_0-\theta)}{L\cos\theta_0+L\sin\theta_0}\,dH(\theta)+\int_{\theta_0}^{\pi/2}\frac{L\sin(\theta-\theta_0)}{L\cos\theta_0+L\sin\theta_0}\,dH(\theta).$$

If θ is uniformly distributed on $[0, \pi/2]$, then $dH(\theta)=d\theta/(\pi/2)$ and $\mu=2/\pi$ as developed in the previous section for isotropic lines.

The number of cars N_L is the number of cars with $-L\sin\theta_0<p<L\cos\theta_0$. Because of our Poisson assumption, N_L is Poisson with parameter $\lambda(L\cos\theta_0+L\sin\theta_0)$. A random member of this set has probability μ of actually overtaking or being overtaken and so the number of such overtaking occurrences is Poisson with parameter (where λ is the intensity of the process of cars entering the highway)

(3.77)

$$\mu\lambda L(\cos\theta_0+\sin\theta_0)=\lambda L\left\{\int_0^{\theta_0}\sin(\theta_0-\theta)\,dH(\theta)+\int_{\theta_0}^{\pi/2}\sin(\theta-\theta_0)\,dH(\theta)\right\}.$$

Since $v=\tan\theta$, we label $G(v)$ as the distribution of the velocity and replace $dH(\theta)$ with $dG(v)$. Also

$$\sin(\theta_0-\theta)=\frac{\sin\theta_0}{(1+v^2)^{1/2}}-\frac{v\cos\theta_0}{(1+v^2)^{1/2}} \tag{3.78}$$

and

$$\sin(\theta-\theta_0)=\frac{v\cos\theta_0}{(1+v^2)^{1/2}}-\frac{\sin\theta_0}{(1+v^2)^{1/2}}.$$

Thus,

$$\begin{aligned}\mu(\cos\theta_0+\sin\theta_0)&=\int_0^{\tan\theta_0}\frac{\sin\theta_0-v\cos\theta_0}{(1+v^2)^{1/2}}\,dH(v)\\&\quad+\int_{\tan\theta_0}^{\infty}\frac{v\cos\theta_0-\sin\theta_0}{(1+v^2)^{1/2}}\,dG(v),\end{aligned} \tag{3.79}$$

$$\mu(\cos\theta_0+\sin\theta_0)=\cos\theta_0\left[\int_0^{v_0}\frac{v_0-v}{(1+v^2)^{1/2}}\,dG(v)+\int_0^{\infty}\frac{v-v_0}{(1+v^2)^{1/2}}\,dG(v)\right].$$

Therefore the intensity of the Poisson process generated by faster cars overtaking the observer car is

$$\lambda'=\lambda L\cos\theta_0\int_{v_0}^{\infty}\frac{v-v_0}{(1+v^2)^{1/2}}\,dG(v), \tag{3.80}$$

and the intensity of the Poisson process generated by the observer car overtaking slower cars is

$$\lambda''=\lambda L\cos\theta_0\int_0^{v_0}\frac{v_0-v}{(1+v^2)^{1/2}}\,dG(v). \tag{3.81}$$

From this basic result arising from a merger of random lines in the plane and nonhomogeneous Poisson processes, all the models developed by Rényi, Weiss and Herman, and others can be derived as demonstrated in Solomon and Wang (1972).

Additional results for anisotropic random lines. In a previous section we found the density for lines in the plane which is invariant under translations and rotations of the coordinate axes. We have just seen for the highway model a much larger class of random lines, whose distribution is required to be invariant only under translations of the coordinate axes. Let us pursue this larger class in other contexts. Three examples of random lines are shown in Fig. 3.8. Figure 3.8(c) shows isotropically distributed random lines. In Fig. 3.8(a) and (b) the random lines are clearly anisotropic (i.e., their orientations are not uniformly distributed).

We now examine the measure of lines intersecting a set. Much of the work that follows is due to Dufour (1972).

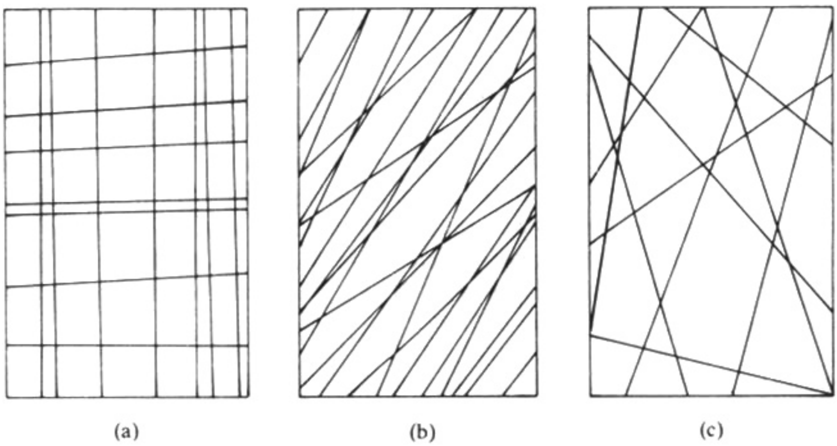


FIG. 3.8

As before we describe the position of a line in terms of the coordinates (p, θ), where p is the signed distance from the origin to the line ($p < 0$ if the line passes beneath the origin), and θ is the angle between the x-axis and the perpendicular to the line. (See Fig. 3.9.) Thus the position of a random line is determined by the

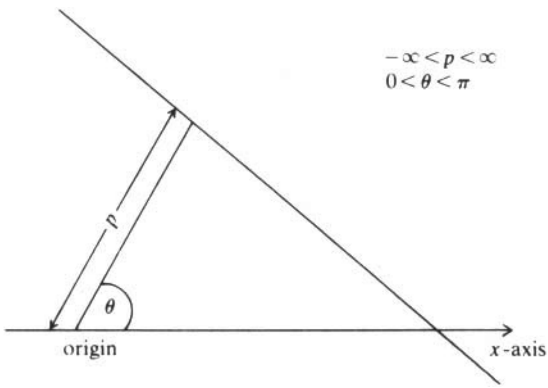


FIG. 3.9

values of the random variables p and θ. We denote by $F(\theta)$ the cumulative distribution function of θ, and let $\int dF(\theta)$ denote integration over the probability density corresponding to the distribution function $F(\theta)$ when $F(\theta)$ is absolutely continuous, summation over the mass function when $F(\theta)$ is discrete, and integration over the singular density when $F(\theta)$ is singular.

Our translation-invariance assumption implies that the orientation of a random line is independent of its distance from the origin. The translation-invariance assumption also implies that for any given θ the density of p is dp. Thus the joint density of p and θ is $dp\, dF(\theta)$ and the measure $m(W)$ of all positions of a random line satisfying some condition W can be written

$$m(W)=\int\int dp\, dF(\theta) \tag{3.82}$$

where the integration covers all positions of the random line such that W is satisfied. Note that in the isotropic case this measure becomes $(1/\pi)\, dp\, d\theta$, which differs by a factor of $1/\pi$ from the measure usually discussed in integral geometry.

At this point we should introduce some basic equations, which we shall use repeatedly throughout this discussion of random lines. Let dG denote the density for the positions of a random geometrical figure K_α (e.g., a random line). Denote by $m(W)$ the measure of all positions of K_α such that some condition W is satisfied, i.e.,

$$m(W)=\int_W dG. \tag{3.83}$$

If X is a random variable depending on the position of K_α, the mean value of X given that K_α satisfies W is

$$E(X\mid W)=\frac{\int_W X\, dG}{m(W)} \tag{3.84}$$

where the integration is taken over all positions of K_α such that the condition W is satisfied. In particular, if we have two conditions W_1 and W_0 such that W_1 implies W_0, then if we define

$$X=\begin{cases}1 & \text{if } W_1 \text{ is satisfied,}\\ 0 & \text{otherwise,}\end{cases}$$

we find that the probability that K_α satisfies W_1, given that it satisfies W_0, is

$$P[W_1\mid W_0]=\frac{m(W_1)}{m(W_0)}. \tag{3.85}$$

Returning to random lines in the plane, let us calculate the measure $m(K)$ of all positions of a random line such that it intersects a (measurable) set K in the plane. Define the thickness $T(\theta)$ of the set K in the direction θ to be the length of the projection of K onto a line with direction θ. (See Fig. 3.10.) Then the integral $\int dp$

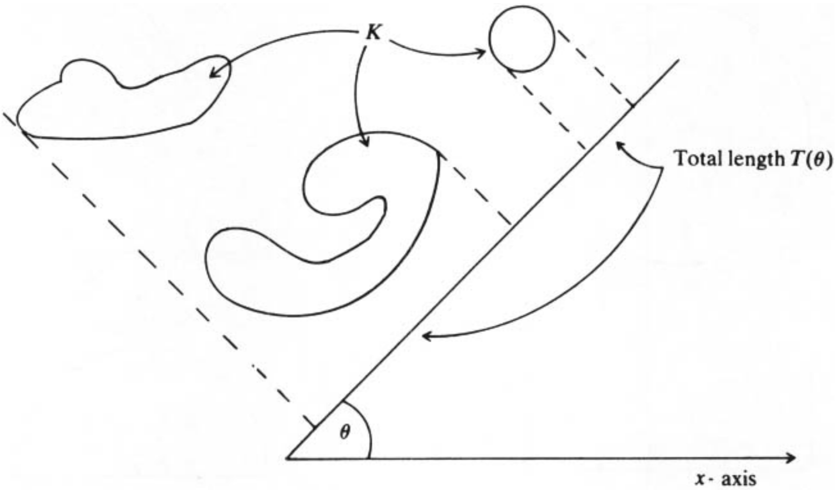


FIG. 3.10

of all positions of a line with perpendicular direction θ such that it intersects the set K is $T(\theta)$. Thus

$$m(K) = \int T(\theta)\, dF(\theta) = E(T) \tag{3.86}$$

where the notation $E(T)$ denotes the mean thickness of the set K in a direction perpendicular to a random line.

Let us consider some simple examples of sets K and distributions $F(\theta)$. First we give expressions for the thickness $T(\theta)$ in the direction of five simple convex regions for which $T(\theta)$ is easily obtained geometrically.

$$\text{(3.87)}\quad \begin{array}{ll} \text{(a)}\quad T_0(\theta) = 2r_0, & \text{(b)}\quad T_0(\theta) = x\,|\cos(\theta-\varphi)|, \\ \text{(c)}\quad T_0(\theta) = s_1|\cos\theta| + s_2|\sin\theta|, & \text{(d)}\quad T_0(\theta) = r(1+|\cos\theta|), \end{array}$$

$$\text{(e)}\quad T_0(\theta) = \begin{cases} s\cos\theta, & 0 \leqq \theta \leqq \dfrac{\pi}{6},\quad \dfrac{5\pi}{6} \leqq \theta \leqq \pi, \\ s\cos\left(\theta - \dfrac{\pi}{3}\right), & \dfrac{\pi}{6} \leqq \theta \leqq \dfrac{\pi}{2}, \\ s\cos\left(\theta - \dfrac{2\pi}{3}\right), & \dfrac{\pi}{2} \leqq \theta \leqq \dfrac{5\pi}{6}. \end{cases}$$

See Fig. 3.11 (a)–(e) for diagrams.

For the circle of radius r, the thickness is a constant, $2r$. Thus the mean thickness is $2r$, and the measure $m(K)$ of the positions of a random line intersecting the circle is $2r$ regardless of the distribution $F(\theta)$ of the orientation of the random line.

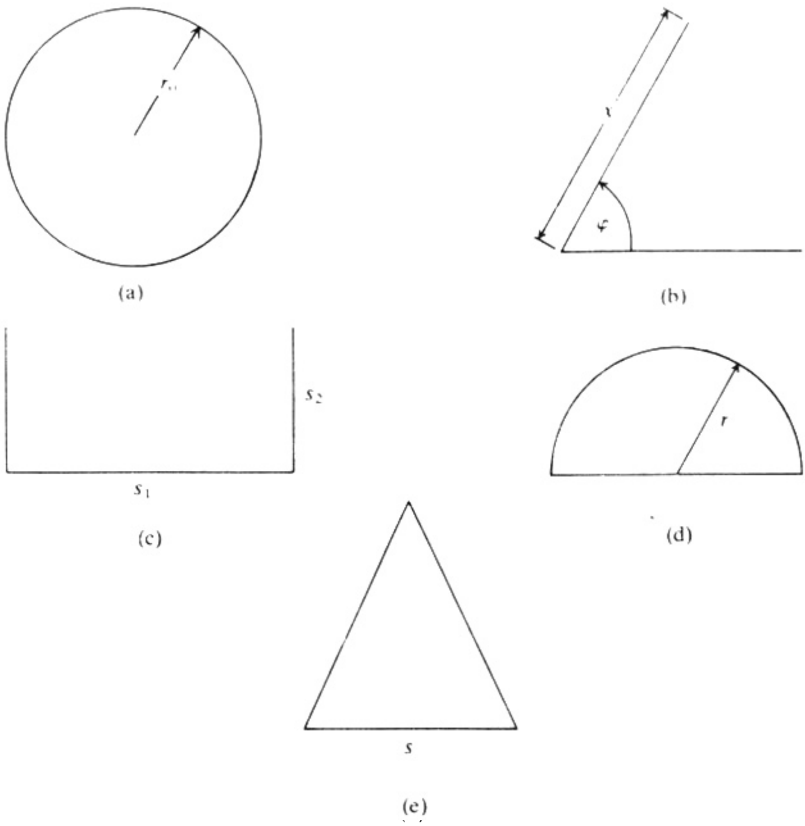


FIG. 3.11

Note that for a connected set that is not convex, the thickness $T(\theta)$ is the same as the thickness of the convex hull of the set. Thus, for example, the thickness $T(\theta)$ of two adjoining line segments of length s making an angle of $\pi/3$ is the same as the $T(\theta)$ given above for the equilateral triangle.

As an example of $F(\theta)$, consider the probability distribution

$$P[\theta=0]=\frac{1}{2}, \qquad P\left[\theta=\frac{\pi}{2}\right]=\frac{1}{2}. \tag{3.88}$$

With this distribution, the mean thickness of a half circle of radius r, for example, is

$$E(T)=\frac{1}{2}[r(1+\cos 0)]+\frac{1}{2}[r(1+\cos \pi/2)]=\frac{3r}{2}. \tag{3.89}$$

As a second example of $F(\theta)$, let us consider the isotropic distribution

$$dF(\theta)=\frac{1}{\pi}\,d\theta, \qquad 0\leqq\theta\leqq\pi. \tag{3.90}$$

Suppose K is a connected set. A line intersects K if and only if it intersects the convex hull of K. In a previous section we showed that if $G \cup K$ is the set of lines intersecting a convex region with perimeter L,

$$\int_G \int_K dp\, d\theta = L. \tag{3.91}$$

Since the integral $\int dp$ over all lines with perpendicular direction θ is just $T(\theta)$, the above relation can be rewritten

$$\int_0^{\pi} T(\theta)\, d\theta = L. \tag{3.92}$$

Thus with respect to an isotropic random line. the mean thickness of a convex region of perimeter L is just

$$E(T) = L/\pi. \tag{3.93}$$

Hence, for an arbitrary connected set K on the plane, $m(K)$ is π^{-1} times the perimeter of the convex hull of the set.

We shall now obtain expressions for the probability that a random line intersects a set K_1 given that it intersects another set K_0. It is necessary that we consider conditional probabilities of this sort since the unconditional probability that a line, random over the whole plane, will intersect any bounded set K_1 is clearly zero. We shall consider three cases: one where K_1 is a subset of K_0, one where K_0 and K_1 have a nonnull intersection, and one where K_0 and K_1 are disjoint.

For the case where K_1 is a subset of K_0, we need only substitute our expressions for the measure $m(K)$ to obtain the following:

A random line intersects a measurable set K_0. The probability that it also intersects a subset K_1 of K_0 is

$$P[\text{line hits } K_1 \mid \text{hits } K_0] = \frac{E(T_1)}{E(T_0)} \tag{3.94}$$

where $E(T_0)$ and $E(T_1)$ are the mean thicknesses of K_0 and K_1, i.e.,

$$E(T_i) = \int_0^{\pi} T_i(\theta)\, dF(\theta), \qquad i = 0, 1. \tag{3.95}$$

An isotropic random line intersects a connected region K_0. It intersects a connected subset K_1 with probability

$$P[\text{line hits } K_1 \mid \text{hits } K_0] = \frac{L_1^*}{L_0^*} \tag{3.96}$$

where L_0^* and L_1^* are the perimeters of the convex hulls of K_0 and K_1, respectively. In particular. if K_0 and K_1 are convex with perimeters L_0 and L_1,

$$P[\text{line hits } K_1 \mid \text{hits } K_0] = \frac{L_1}{L_0}. \tag{3.97}$$

Let us now consider the case where K_1 and K_0 have a nonnull intersection, but K_1 is no longer necessarily contained in K_0. We shall assume further that K_0 and K_1 are connected sets, since without this assumption the problem seems to be intractable. Given that a random line hits K_0, the probability that it also hits K_1 is

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{m(K_0 \text{ and } K_1)}{m(K_0)} \tag{3.98}$$

where $m(K_0 \text{ and } K_1)$ denotes the measure of all lines intersecting both K_0 and K_1. The measure $m(K_0 \text{ or } K_1)$ of lines hitting K_0 or K_1 can be written

$$m(K_0 \text{ or } K_1) = m(K_0) + m(K_1) - m(K_0 \text{ and } K_1). \tag{3.99}$$

Thus

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{m(K_0) + m(K_1) - m(K_0 \text{ or } K_1)}{m(K_0)}, \tag{3.100}$$

and we have the following:

Let K_0 and K_1 be the intersecting connected sets on the plane. A random line which intersects K_0 also intersects K_1 with probability

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{E(T_0) + E(T_1) - E(T_{01})}{E(T_0)} \tag{3.101}$$

where $E(T_0)$, $E(T_1)$ and $E(T_{01})$ denote the mean thicknesses of K_0, K_1 and $K_0 \cup K_1$, respectively.

Note that when $K_1 \subset K_0$, $E(T_{01})$ is $E(T_0)$ and the probability given in this theorem reduces to that given before.

For an isotropic random line

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{L_0^* + L_1^* - L_{01}^*}{L_0^*} \tag{3.102}$$

where L_0^*, L_1^*, and L_{01}^* are the perimeters of the convex hulls of K_0, K_1 and $K_0 \cup K_1$, respectively.

Now consider the case where K_0 and K_1 are disjoint, connected sets. A line intersects both K_0 and K_1 if and only if it intersects both of the shaded regions $(K_0 \cup Q)^*$ and $(K_1 \cup Q)^*$ pictured in Fig. 3.12.

The point Q is the intersection of two lines just touching the edges of K_0 and K_1 and crossing between them. The sets $(K_0 \cup Q)^*$ and $(K_1 \cup Q)^*$ are the convex hulls of $K_0 \cup Q$ and $K_1 \cup Q$, respectively. Thus

$$\begin{aligned} m(K_0 \text{ and } K_1) &= m((K_0 \cup Q)^* \text{ and } (K_1 \cup Q)^*) \\ &= m((K_0 \cup Q)^*) + m((K_1 \cup Q)^*) \\ &\quad - m((K_0 \cup Q)^* \text{ or } (K_1 \cup Q)^*). \end{aligned} \tag{3.103}$$

Since $(K_0 \cup Q)^*$ and $(K_1 \cup Q)^*$ intersect, a line hits $(K_0 \cup Q)^*$ or $(K_1 \cup Q)^*$ if

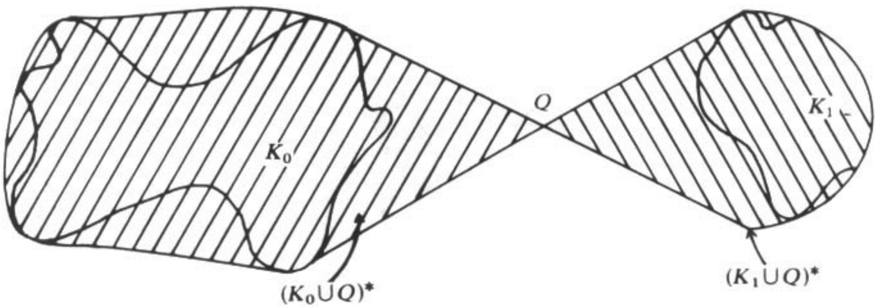


FIG. 3.12

and only if it hits the convex hull of $(K_0 \cup Q)^* \cup (K_1 \cup Q)^*$ which is just $(K_0 \cup K_1)^*$, the convex hull of $K_0 \cup K_1$. Thus

$$m(K_0 \text{ and } K_1) = m((K_0 \cup Q)^*) + m((K_1 \cup Q)^*) - m((K_0 \cup K_1)^*) \tag{3.104}$$

and we have the following.

Let K_0 and K_1 be disjoint, connected sets on the plane. A random line which intersects K_0 also intersects K_1 with probability

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{E(T_{0Q}^*) + E(T_{1Q}^*) - E(T_{01}^*)}{E(T_0)} \tag{3.105}$$

where $E(T_0)$ is the mean thickness of K_0 and $E(T_{0Q}^*)$, $E(T_{1Q}^*)$, and $E(T_{01}^*)$ are the mean thicknesses of the convex hulls of $K_0 \cup Q$, $K_1 \cup Q$ and $K_0 \cup K_1$, respectively, where Q is the intersection of two lines just touching the edges of K_0 and K_1 and crossing between them.

For an isotropic random line

$$P[\text{line hits } K_1 \,|\, \text{hits } K_0] = \frac{L_{0Q}^* + L_{1Q}^* - L_{01}^*}{L_0^*} \tag{3.106}$$

where L_{0Q}^*, L_{1Q}^*, L_{01}^* and L_0^* are the perimeters of the convex hulls of $K_0 \cup Q$, $K_1 \cup Q$, $K_0 \cup K_1$ and K_0, respectively.

Consider the mean length of intersection of a random line with a set. Suppose a random line intersects a measurable set K on the plane. The mean value of the length C of line within K is

$$\begin{aligned} E(C) &= \frac{\iint C\,dp\,dF(\theta)}{m(K)} \\ &= \frac{\iint C\,dp\,dF(\theta)}{E(T)}. \end{aligned} \tag{3.107}$$

The integral $\int C\,dp$ is just the area A of K. (This fact is easy to see by breaking K into thin parallel strips as in Fig. 3.13 and adding up the areas of the strips.)

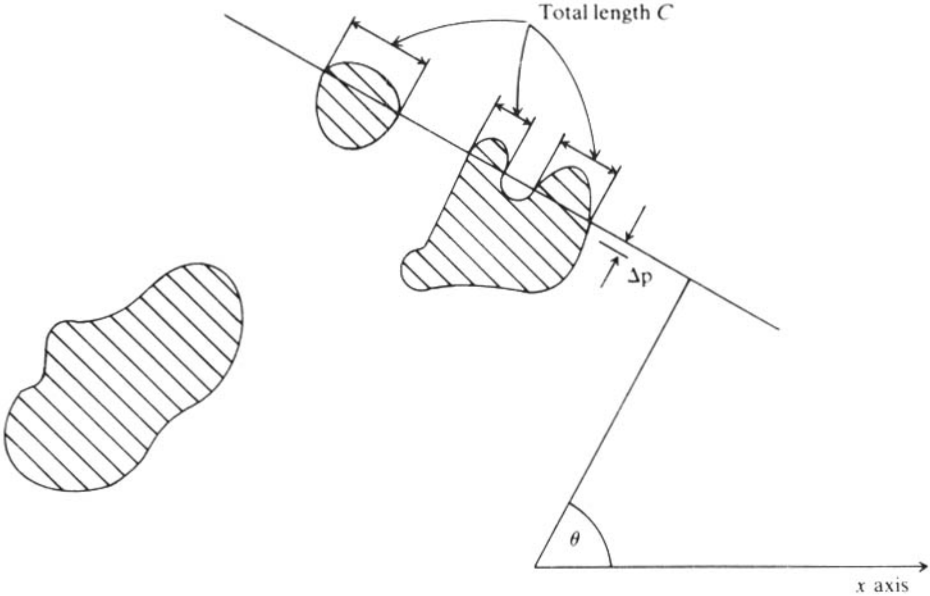


FIG. 3.13

Hence

$$E(C) = \frac{\int A \, dF(\theta)}{E(T)} = \frac{A}{E(T)}. \tag{3.108}$$

If K is a connected region and the random line is isotropically distributed,

$$E(C) = \frac{\pi A}{L^*} \tag{3.109}$$

where L^* is the perimeter of the convex hull of E. In particular, when K is convex with perimeter L we have

$$E(C) = \frac{\pi A}{L}, \tag{3.110}$$

one of the results of Crofton.

Now consider random lines intersecting within a region. We shall now find the probability that the two independent random lines, both intersecting a region K, intersect within K. Suppose the line segments formed within K by the first line have total length C_1 and perpendicular direction θ_1. A second random line hitting K intersects these line segments with probability

$$P[\text{Intersect in } K \mid C_1, \theta_1] = \frac{E_2(T_1)}{E_2(T)} \tag{3.111}$$

where $T_1(\theta_2)$ is the thickness of the intersection of the first line with K, and the

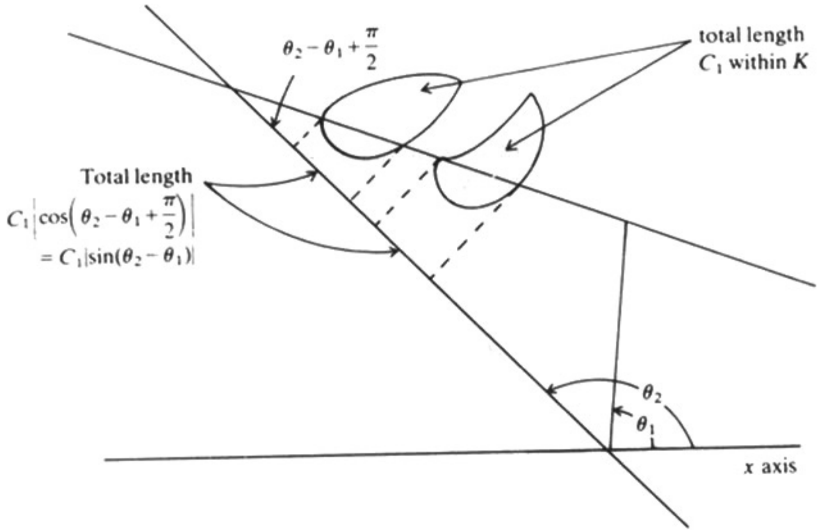


FIG. 3.14

notation E_2 indicates expectation with respect to the distribution $F_2(\theta)$ of the orientation of the second line. From Fig. 3.14 we see that

(3.112)
$$E_2(T_1) = C_1 E_2(|\sin(\theta_2 - \theta_1)|)$$

so that

(3.113)
$$P[\text{Intersect in } K \mid C_1, \theta_1] = \frac{C_1 E_2(|\sin(\theta_2 - \theta_1)|)}{E_2(T_0)}.$$

Integrating over all possible positions of the first line, we get

(3.114)
$$P[\text{Intersect in } K] = \frac{E[C_1 E_2(|\sin(\theta_2 - \theta_1)|)]}{E_2(T)}.$$

The numerator is

$$E[C_1 E_2[|\sin(\theta_2 - \theta_1)|]] = \int_0^\pi \int C_1 \int_0^\pi |\sin(\theta_2 - \theta_1)|\, dF_2(\theta_2)\, dp_1\, dF_1(\theta_1)$$

(3.115)
$$= \frac{\int_0^\pi [\int C_1\, dp_1] \int_0^\pi |\sin(\theta_2 - \theta_1)|\, dF_2(\theta_2)\, dF_1(\theta_1)}{E_1(T)}$$

$$= \frac{A \int_0^\pi \int_0^\pi |\sin(\theta_2 - \theta_1)|\, dF_2(\theta_2)\, dF_1(\theta_1)}{E_1(T)}.$$

Thus we may write two independent random lines whose perpendicular directions have the distribution functions F_1 and F_2 and intersect a region K. They

intersect each other within K with probability

$$P[\text{Intersect in } K] = \frac{A \int_0^\pi \int_0^\pi |\sin(\theta_1 - \theta_2)|\, dF_1(\theta_1)\, dF_2(\theta_2)}{E_1(T)E_2(T)} \tag{3.116}$$

where $E_1(T)$ and $E_2(T)$ are the mean thicknesses of K with respect to the distributions F_1 and F_2, respectively.

Since

$$\frac{1}{\pi}\int_0^\pi |\sin(\theta_1 - \theta_2)|\, d\theta_1 = \frac{2}{\pi}, \tag{3.117}$$

the factor $\int_0^\pi \int_0^\pi |\sin(\theta_1 - \theta_2)|\, dF_1(\theta_1)\, dF_2(\theta_2)$ is just $2/\pi$ if either of the lines is isotropically distributed.

If one of the lines is isotropically distributed, say the one corresponding to subscript 1,

$$P[\text{Intersect in } K] = \frac{2A}{\pi E_1(T)E_2(T)}. \tag{3.118}$$

If, in addition, K is a connected region,

$$P[\text{Intersect in } K] = \frac{2A}{L^* E_2(T)} \tag{3.119}$$

where L^* is the perimeter of the convex hull of K. If both lines are isotropic and K is connected,

$$P[\text{Intersect in } K] = \frac{2\pi A}{L^{*2}} \tag{3.120}$$

which, when K is convex with perimeter L, reduces to

$$P[\text{Intersect in } K] = \frac{2\pi A}{L^2}, \tag{3.121}$$

another of Crofton's well-known results.

We now examine angles between intersecting random lines. For any two random lines with perpendicular directions θ_1 and θ_2, the angles between the lines are $|\theta_1 - \theta_2|$ and $\pi - |\theta_1 - \theta_2|$, whose probability distribution can be calculated from

$$P[|\theta_1 - \theta_2| \leqq a] = \iint_{|\theta_1 - \theta_2| \leqq a} dF_1(\theta_1)\, dF_2(\theta_2). \tag{3.122}$$

Thus, for example, if either line is isotropic, the angle between the lines is uniformly distributed from 0 to π.

On the other hand, suppose we pick two random lines which are known to intersect within some finite region K of the type considered before. The distribution of the angle between these lines is not the same as the distribution of

the angle formed between two particular random lines. Let I_{12} denote the event that the two random lines intersect within K. Then

$$P[|\theta_1-\theta_2|<a\,|\,I_{12}]=\frac{P[|\theta_1-\theta_2|<a\,|\,I_{12}]}{P[I_{12}]} \tag{3.123}$$
$$=\frac{\iint_{|\theta_1-\theta_2|<a}P[I_{12}\,|\,\theta_1,\theta_2]\,dF_1(\theta_1)\,dF_2(\theta)}{P[I_{12}]}.$$

From our previous result, we have

$$P[I_{12}\,|\,\theta_1,\theta_2]=\frac{A|\sin(\theta_1-\theta_2)|}{E_1(T)E_2(T)} \tag{3.124}$$

and

$$P[I_{12}]=\frac{A\int_0^\pi\int_0^\pi|\sin(\theta_1-\theta_2)|\,dF_1(\theta_1)\,dF_2(\theta_2)}{E_1(T)E_2(T)} \tag{3.125}$$

so that

$$P[|\theta_1-\theta_2|<a\,|\,I_{12}]=\frac{\iint_{|\theta_1-\theta_2|<a}|\sin(\theta_1-\theta_2)|\,dF_1(\theta_1)\,dF_2(\theta_2)}{\int_0^\pi\int_0^\pi|\sin(\theta_1-\theta_2)|\,dF_1(\theta_1)\,dF_2(\theta_2)}. \tag{3.126}$$

If either of the lines is isotropically distributed, say, the second

$$\frac{1}{\pi}\int_0^\pi|\sin(\theta_1-\theta_2)|\,d\theta_2=\frac{2}{\pi} \tag{3.127}$$

and

$$\frac{1}{\pi}\iint_{|\theta_1-\theta_2|<a}|\sin(\theta_1-\theta_2)|\,dF_1(\theta_1)\,d\theta_2=\frac{1}{\pi}\iint_{|\varphi|<a}|\sin\varphi|\,d\varphi\,dF_1(\theta_1) \tag{3.128}$$
$$=\frac{1}{\pi}\int_{|\varphi|<a}|\sin\varphi|\,d\varphi.$$

Thus for $0\leqq a\leqq\pi$

$$P[|\theta_1-\theta_2|<a\,|\,I_{12}]=\frac{1}{2}\int_{|\varphi|<a}\sin\varphi\,d\varphi. \tag{3.129}$$

Therefore if two random lines intersect within some finite region in the plane, the angle φ formed at their intersection has the probability density $\frac{1}{2}\sin\varphi$ for $0\leqq\varphi\leqq\pi$, if either of the intersecting lines is isotropic.

Now consider an infinite collection of random lines, independently and identically distributed over the plane. Define the intensity τ of the collection of random lines to be the mean number of lines intersecting a circle of unit diameter. We shall find the probability distribution of the number of lines

intersecting a set K made up of the areas within a finite number of closed curves on the plane.

Let $C_1, C_2, \cdots$ be concentric circles, the circle C_j having diameter j. Let N_j denote the number of random lines intersecting C_j, and define

$$X_j = N_j - N_{j-1}, \tag{3.130}$$

that is, X_j is the number of random lines hitting C_j but not C_{j-1}. Since the random lines are independently distributed, the random variables $X_1, X_2, \cdots$ are independent. Since the mean thickness $E(T)$ of a circle is its diameter, a random line which hits C_n also intersects C_j within C_n $(j \leqq n)$ with probability j/n. Thus the probability that a line hitting C_n hits C_j but not C_{j-1} is

$$\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n} \tag{3.131}$$

and

$$P[X_j = k | N_n \text{ lines hit } C_n] = \binom{N_n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{N_n-k}. \tag{3.132}$$

Since this does not depend on j, the unconditional probability $P[X_j = k]$ does not depend on j. In other words, the variables $X_1, X_2, \cdots, X_n$ are identically distributed. Since the X_j are independent and identically distributed, the law of large numbers yields

$$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow{\text{P}} \tau \quad \text{as } n \to \infty, \tag{3.133}$$

that is,

$$\frac{N_n}{n} \xrightarrow{\text{P}} \tau \quad \text{as } n \to \infty \tag{3.134}$$

where the notation $\xrightarrow{\text{P}}$ indicates convergence in probability.

Now consider any set K made up of the areas within a finite number of closed curves on the plane. Let K be contained in a large circle C_n of diameter n. A random line which hits C_n also hits K with probability $E(T)/n$. Thus the probability distribution of N_k, the number of lines intersecting K, given N_n, the number hitting C_n, is

$$\begin{aligned} P[N_k = k \,|\, N_n] &= \binom{N_n}{k}\left[\frac{E(T)}{n}\right]^k\left[1-\frac{E(T)}{n}\right]^{N_k-k} \\ &= \frac{1}{k!}\,\frac{N_n}{n}\,\frac{(N_n-1)}{n}\cdots\frac{(N-k+1)}{n} \\ &\quad \cdot [E(T)]^k\left[\left(1-\frac{E(T)}{n}\right)^n\right]^{(N_n-k)/n}. \end{aligned} \tag{3.135}$$

Since $P[N_k - k \mid N_n]$ is a continuous function of N_n/n, the fact that $N_n/n \xrightarrow{p} \tau$ as $n \to \infty$ implies

$$(3.136) \qquad P[N_K = k \mid N_n] \xrightarrow{p} \frac{\tau^k}{k!}[E(T)]^k e^{-\tau E(T)}.$$

Since $P[N_k = k \mid N_n]$ is bounded, the above convergence implies

$$(3.137) \qquad E(P[N_k = k \mid N_n]) \to \frac{e^{-\tau E(T)}[\tau E(T)]^k}{k!}$$

as $n \to \infty$. This, in turn, implies

$$(3.138) \qquad P[N_k = k] = \frac{e^{-\tau E(T)}[\tau E(T)]^k}{k!}$$

and so N_k has a Poisson distribution with mean $\tau E(T)$, where $E(T)$ is the mean thickness of the set K and τ is the intensity of the random lines.

Moreover the intersections of random lines with a line having direction φ are the events of a Poisson process with mean rate $\tau E(|\cos(\theta - \varphi)|)$. In particular, the distance x from any point on the line to the first intersection by a random line has the probability density $\tau E(|\cos(\theta - \varphi)|)\, e^{-\tau E(|\cos(\theta-\varphi)|)x}$ for $x > 0$. This result has already been developed in the highway model.

For the isotropic case, we have the following simplifications. The number of isotropic random lines intersecting a region K within some closed curve has a Poisson distribution with mean $\tau L^*/\pi$. Furthermore, the intersections of the isotropic random lines with any particular line are events of a Poisson process with mean rate $2\tau/\pi$. Thus the distance x along the line from an arbitrary point to the first intersection has the density $(2\tau/\pi)\, e^{-2\tau x/\pi}$ for $x \geqq 0$. This was also derived previously.

Now consider the points of intersection of the random lines in a Poisson field of random lines on the plane. Due to the translation-invariance property, the points of intersection are uniformly distributed over the plane. The mean number of these points of intersection within any region K can be found as follows.

Suppose N_K lines intersect the region K. Let M_K denote the number of intersections within K of the N_K "chords" formed by the lines intersecting K. (Since K may be nonconvex, each "chord" may consist of several colinear line segments.) Define

$$\alpha_{jk} = \begin{cases} 1 & \text{if the } j\text{th and } k\text{th "chords" intersect in } K, \\ 0 & \text{otherwise.} \end{cases}$$

Then M_K can be written

$$(3.139) \qquad M_K = \sum_{j=1}^{N_K} \sum_{k=j+1}^{N_K} \alpha_{jk}$$

and

$$(3.140) \qquad E(M_K \mid N_K) = \frac{1}{2} N_K (N_K - 1) E(\alpha_{jk}).$$

Since, by the previous result, N_K has a Poisson distribution with mean $\tau E(T)$,

$$E(M_K)=\frac{1}{2}[\tau E(T)]^2 E(\alpha_{jk}). \tag{3.141}$$

Moreover the probability that two lines intersect within K, given that they hit K, is

$$E(\alpha_{jk})=\frac{A\int_0^\pi\int_0^\pi|\sin(\theta_1-\theta_2)|\,dF(\theta_1)\,dF(\theta_2)}{[E(T)]^2}. \tag{3.142}$$

Combining these expressions for $E(M_K)$ and $E(\alpha_{jk})$, we get the points of intersection of the lines of a Poisson field of random lines on the plane are uniformly distributed over the plane. The mean number of such intersections within a region K with area A is

$$E(M_K)=\frac{1}{2}\tau^2 A\int_0^\pi\int_0^\pi|\sin(\theta_1-\theta_2)|\,dF(\theta_1)\,dF(\theta_2) \tag{3.143}$$

where τ is the intensity of the Poisson field (i.e., the mean number of random lines intersecting a circle of unit diameter).

The mean number of points of intersection of isotropic random lines within K is

$$E(M_K)=\frac{\tau^2 A}{\pi}. \tag{3.144}$$

The mean number of intersections outside of K among lines intersecting K can be obtained. The total number of intersections among the N_K chords hitting K is $\frac{1}{2}N_K(N_K-1)$. Thus the number of intersections outside of K is

$$E(M_K\,|\,N_K)=\frac{1}{2}N_K(N_K-1)[P[\theta_i\neq\theta_j]-E(\alpha_{ij})] \tag{3.145}$$

and

$$E(M_K)=\frac{1}{2}[\tau E(T)]^2[P[\theta_i\neq\theta_j]-E(\alpha_{ij})], \tag{3.146}$$

or

$$E(M_K)=\frac{1}{2}\tau^2\left[[E(T)]^2P[\theta_i\neq\theta_j]-A\int_0^\pi\int_0^\pi|\sin(\theta_i-\theta_j)|\,dF(\theta_i)\,dF(\theta_j)\right]$$

where $P[\theta_i\neq\theta_j]$ is the probability that two random lines are not parallel.

CHAPTER 4

Covering a Circle Circumference and a Sphere Surface

In his turn of the century book Whitworth (1897) was one of the first to consider problems relating to the coverage of a circle by random arcs. Suppose we place n arcs of size a uniformly and independently on the circumference of a circle. Then Whitworth's answer to exercise 667 in his volume can be interpreted as the probability that the part of the circumference that is not covered by an arc consists of exactly $n-r$ connected components, that is, the pattern of arcs leaves $n-r$ gaps on the circumference. He restricted himself, by the comment in exercise 666 "say $c-na=ma$" where c is the size of the circumference of the circle, to the case in which $na<c$.

This restriction was apparently overlooked by F. Garwood (1940) who incorrectly credits Whitworth with the evaluation of the probability of complete coverage of the circle by these arcs. Complete coverage corresponds to the case $r=n$. However, the restriction $na<c$ implies that there are not enough arcs to cover the circle completely. Thus Whitworth's result excludes the case of interest, namely probability of coverage when it is possible to cover. Indeed, Whitworth's formula with $r=n$ sums to zero identically for any value of a, as can be seen using elementary theory of finite differences, and thus cannot be correct when $na>c$. Darling (1953) and Shepp (1972), apparently because of Garwood's remarks, also mistakenly attribute this result to Whitworth.

M. Baticle (1935) comes very close to evaluating this coverage probability while considering a problem of partitioning an interval, but he does not state when one should stop summing his series. To be correct, we must stop at the $[c/a]+1$ term, where brackets indicate the greatest integer less than or equal to c/a. If one sums completely, as he indicated, one gets zero, as was the case for Whitworth's formula.

W. L. Stevens (1939) thus appears to be the first to evaluate the probability that n random arcs of size a cover a unit circumference completely and he provides an ingenious solution. This probability, for a circle of circumference one, is

$$(4.1)\qquad 1-\binom{n}{1}(1-a)^{n-1}+\binom{n}{2}(1-2a)^{n-1}-\cdots+(-1)^k\binom{n}{k}(1-ka)^{n-1}$$

where $k=[1/a]$, the greatest integer contained in $1/a$. By using a geometrical argument, Stevens also solves generally for the probability that the arcs will

leave exactly l uncovered "gaps" between them. This is shown to be

$$\binom{n}{l}\sum_{j=l}^{k}(-1)^{j-l}\binom{n-l}{j-l}(1-ja)^{n-1}. \tag{4.2}$$

There is a series much like (4.1) derived by R. A. Fisher (1929) as the null distribution of a test statistic for significance in time series analysis. Fisher (1940) noted the similarity between his series and Stevens' result and the link between the two problems.

We now develop the Stevens solution and then add a number of additional results due to Siegel (1977).

Random arcs on the circumference of a circle. Consider a circle of unit circumference. Let us drop n arcs each of length a, $0 < a < 1$, at random onto the circle with circumference equal to 1. We desire the probability that every point of the circle is covered by at least one of the arcs.

We can identify each arc by its clockwise endpoint, and we can assume these n points (corresponding to the n arcs) are uniformly distributed over the circumference of the circle. Without loss of generality we can rotate the circle so that the clockwise endpoint of one arc falls in the twelve o'clock position on the circle. This arc is labeled 1 and the remaining arcs are labeled in the counter-clockwise direction from 2 to n. (See Fig. 4.1.)

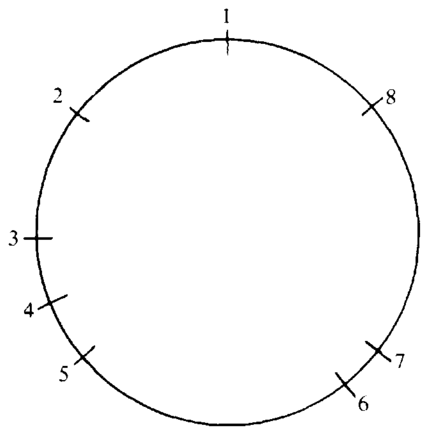


FIG. 4.1

First we shall find the probability $f(1)$ that there is a gap or uncovered portion of the circle after the rth arc, r specified. Let G_r denote the event that there is a gap after the rth arc. Let F_a denote the event that no clockwise endpoint falls in the portion of circle of length a extending in the clockwise direction from point 1. Then there is a one to one correspondence between the configurations of arcs where G_r occurs and the configurations of arcs where F_a occurs. This correspondence can be seen by considering the operator T_r which rotates arcs

$r+1, r+2, \cdots, n$ clockwise through a distance a. For any configuration C of arcs where G_r occurs, $T_r(C)$ is a valid configuration of arcs (i.e., the ordering of the arcs is preserved) where F_a occurs. On the other hand, if C' is any configuration of arcs where F_a occurs, then $T_r^{-1}(C')$ is a valid configuration of arcs where G_r (i.e., a gap after the rth arc) occurs. The effect of the operator T_r is illustrated in Fig. 4.2. This one to one correspondence implies that

$$P[G_r] = P[F_a].$$

But $P[G_r] = f(1)$, namely the probability that there is a gap or uncovered portion of the circle after the rth arc, and $P[F_a] = (1-a)^{n-1}$. Thus we have

$$f(1) = (1-a)^{n-1}. \tag{4.3}$$

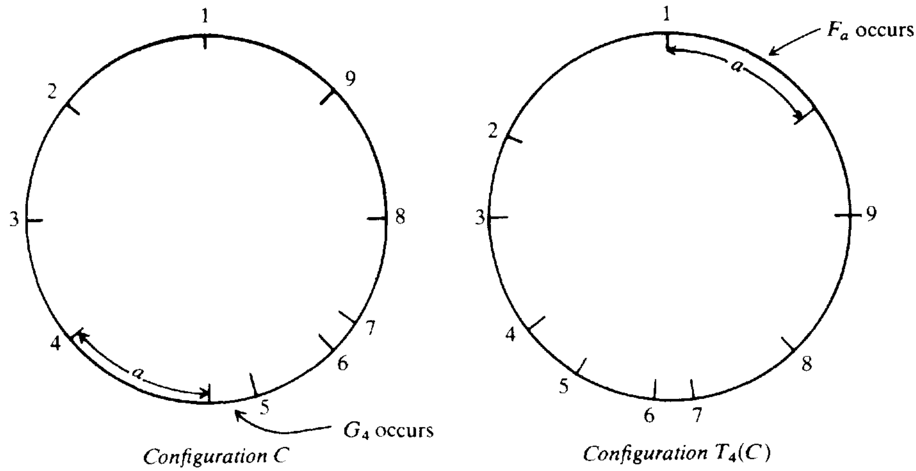


FIG. 4.2

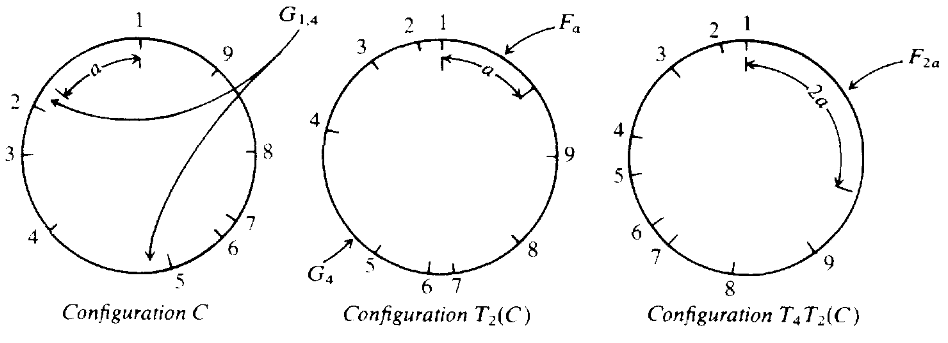


FIG. 4.3

In the same way we can find the probability $f(2)$ that there is a gap directly after arc r_1 and directly after arc r_2. Let G_{r_1,r_2} denote this event. Then the

operator $T_{r_1}T_{r_2}$ provides a one to one correspondence between configurations where G_{r_1,r_2} occurs and configurations where F_{2a} occurs. This correspondence is illustrated in Fig. 4.3. Thus

$$P[G_{r_1,r_2}]=P[F_{2a}].$$

But $P[G_{r_1,r_2}]=f(2)$, namely the probability that there is a gap after the r_1th arc and a gap after the r_2th arc and $P[F_{2a}]=(1-2a)^{n-1}$, so we have

$$f(2)=(1-2a)^{n-1}. \tag{4.4}$$

It is clear that we can use this same procedure to obtain the probability that there are gaps in i specified places. The distance between each pair of arcs enclosing a gap may be reduced by a (by shifting the arcs labeled by higher numbers a distance a clockwise), to produce a configuration with the portion of length ia terminating at 1 containing no clockwise endpoints of arcs, and conversely. Hence, if we let $f(i)$ denote the probability of gaps after i specified arcs (regardless of what happens elsewhere) then

$$f(i)=\begin{cases}(1-ia)^{n-1}, & i\leqq k,\\ 0, & i>k,\end{cases} \tag{4.5}$$

where k is the greatest integer less than $1/a$.

To proceed with the solution we develop a theorem in probability concerning a random n-tuple whose components have two possible outcomes H or T. The components are not independent, but the probability that any h specified components out of the n are H, whatever the remainder, is known to be $f(h)$. We wish to develop formulae for the probability $f(h, t)$ that h specified components are H, t specified components are T and the remaining $n-h-t$ are either H or T.

First we shall evaluate $f(h, 1)$. Let j be the component which is specified to be a T. Consider the set of all outcomes of the random n-tuple in which h specified components are H's. This set has probability $f(h)$. Delete from this set all outcomes in which there is a T in component j, i.e., all outcomes with h specified H's and one specified T. The deleted set has probability $f(h, 1)$. The resulting set has H's not only in the original h specified components but also in component j. The probability of the set that results is $f(h+1)$. Thus we have

$$f(h)-f(h, 1)=f(h+1) \tag{4.6a}$$

or

$$f(h, 1)=f(h)-f(h+1)=-\Delta f(h), \tag{4.6b}$$

where Δ denotes the difference operator.

With a similar argument we can obtain the probability $f(h, 2)$. Let j and m be the components that are to be T's. Consider the set of all outcomes in which h specified components are H's and the component j is a T. This set has probability $f(h, 1)$. Delete from this set all outcomes with a T in component m. The deleted set has probability $f(h, 2)$. The members of the resulting set have H's in

the h specified places as well as in component m and a T in component j. Its probability is $f(h+1, 1)$. Thus

$$f(h, 1)-f(h, 2)=f(h+1, 1)$$

or

$$f(h, 2)=f(h, 1)-f(h+1, 1)=-\Delta f(h, 1)=\Delta^2 f(h).$$

This indicates that, in general, for $t \leqq n-h$

$$f(h, t)=(-\Delta)^t f(h)=f(h)-tf(h+1)+\frac{t(t-1)}{2}f(h+2)-\cdots(-1)^t f(h+t). \tag{4.7}$$

This expression can be proved using induction and the same argument as above. Assume it is true for all h and a particular t, any t_0. Consider the set of all outcomes in which h specified components are H's and t_0 specified components are T's. This set has probability $f(h, t_0)$. Let j be any component whose value is not specified. Delete from the above set all outcomes with a T in component j. The deleted set has probability $f(h, t_0+1)$. Each member of the resulting set has H's in the h specified places as well as in component j and T's in the t_0 specified places for T's. This set has probability $f(h+1, t_0)$. Thus

$$f(h, t_0)-f(h, t_0+1)=f(h+1, t_0) \tag{4.8a}$$

or

$$f(h, t_0+1)=f(h, t_0)-f(h+1, t_0)=-\Delta f(h, t_0)=(-\Delta)^{t_0+1}f(h) \tag{4.8b}$$

and the proof by induction is complete. This can be done also without the use of operators by substituting in the series expansion given above for $f(h, t)$.

In particular the probability that h specified events are n, and the remainder all T, is

$$\begin{aligned} f(h, n-h) &= (-\Delta)^{n-h}f(h) \\ &= f(h)-(n-h)f(h+1) \\ &\quad +\frac{(n-h)(n-h-1)}{2}f(h+2)\cdots(-1)^{n-h}f(h). \end{aligned} \tag{4.9}$$

An easy way to evaluate this expression is to represent the difference operator Δ as

$$\Delta = U-1$$

where U is a unit shift operator; i.e.,

$$Uf(h)=f(h+1) \quad \text{and} \quad U^j f(h)=f(h+j).$$

Then

$$
\begin{aligned}
f(h, n-h) &= (-\Delta)^{n-h} f(h) \\
&= (1-U)^{n-h} f(h) \\
&= \sum_{j=0}^{n-h} \binom{n-h}{j} (-U)^j f(h) \\
&= \sum_{j=0}^{n-h} \binom{n-h}{j} (-1)^j f(h+j).
\end{aligned} \tag{4.10}
$$

Up to this point we have considered only probabilities in which the exact positions of the H's and T's are specified. The probability of h H's and $(n-h)$ T's, regardless of their positions, is obtained by noting that there are $\binom{n}{h}$ ways of selecting h components from the n-tuple. Thus the probability of exactly h H's is

$$
\binom{n}{h} \sum_{j=0}^{n-h} \binom{n-h}{j} (-1)^j f(h+j). \tag{4.11}
$$

We can apply this result to obtain the solution to our problem of random arcs covering the circumference of a circle. Let the alternative H and T in the jth component of the n-tuple refer to the existence or nonexistence, respectively, of a gap after the jth arc. We have already evaluated the probabilities

$$
f(i) = \begin{cases} (1-ia)^{n-1}, & i \leqq k, \\ 0, & i > k, \end{cases}
$$

of gaps in i specified places. Thus we have:

$$
\begin{aligned}
P[\text{No gaps}] &= \sum_{j=0}^{k} \binom{n}{j} (-1)^j (1-ja)^{n-1} \\
&= 1 - n(1-a)^{n-1} + \frac{n(n-1)}{2}(1-2a)^{n-1} - \cdots \\
&\quad \pm \frac{n(n-1)\cdots(n-k+1)}{k!}(1-ka)^{n-1},
\end{aligned} \tag{4.12}
$$

$$
P[\text{One gap}] = n \sum_{j=0}^{k-1} \binom{n-1}{j} (-1)^j (1-[j+1]a)^{n-1}, \tag{4.13}
$$

$$
P[i \text{ gaps}] = \binom{n}{i} \sum_{j=0}^{k-i} \binom{n-i}{j} (-1)^j (1-[j+i]a)^{n-1}, \tag{4.14}
$$

$$
P[k \text{ gaps}] = \binom{n}{k} (1-ka)^{n-1} \tag{4.15}
$$

where k is the greatest integer less than $1/a$.

Thus the probability of covering the circumference with n arcs can be computed from $P[\text{No gaps}]=f(0)$. For example, if $a=\frac{1}{5}$, $f(0)=.4929$ for $n=16$, $f(0)=.5596$ for $n=17$. Thus, we require 17 arcs to be at least 50% sure that the circumference will be covered.

C. Domb (1947) considered a generalization of Stevens' problem. Intervals of length a are placed on R the real line, with left endpoints as the events of a Poisson process. The moments and the distribution of the proportion of the interval $[0, y]$ covered by such segments is calculated, together with the probability of complete coverage, by the use of a Laplace transform argument on a system of integral equations. These results can be transformed to give the corresponding results for the problem of N random arcs of size a on the circumference on a circle, when N has a Poisson distribution. By expanding his coverage probability result in powers of λ, the intensity of the Poisson process, Domb is able to produce Stevens' formula for coverage by n arcs, with n fixed. In principle, a similar expansion would yield the moments and the distribution of that proportion of the circle covered by a fixed number n of random arcs. However, these are not produced by Domb because for the distribution, "the complete solution . . . is very complicated" and in the case of the higher moments, because "they become rather cumbersome." Both the moments and the distribution are produced explicitly, by a method different from Domb's, in Siegel (1977), which we produce shortly.

J. G. Mauldon (1951) generalized Stevens' result in a different way. If we let $Y_1, \cdots, Y_n$ be the gaps between n points placed uniformly and independently on the circle, then $P(\max_{1\leqq i\leqq n} Y_i \leqq a)$ is the probability that n random arcs of size a cover the circle. This is easily seen if we place the clockwise endpoint of one arc at each Y_i. Thus Stevens' result is seen to yield the distribution of the largest Y_i. Mauldon derived the moments and the distribution of the sum of the k largest of $Y_1, \cdots, Y_n$. Thus the case $k=1$ reduces to the circle coverage probability (1).

Let $Y_1, \cdots, Y_n$ be gaps on the circle as in the previous paragraph. D. A. Darling (1953) found a complex contour integral representation of the characteristic function of the random variable $W_n=\sum_{j=1}^n h_j(Y_j)$ for quite arbitrary functions h_j. His method gives a unified treatment to problems arising in the study of infectious diseases, traffic flow, etc., as well as some problems in nonparametric tests for goodness of fit. Among results derived by the use of his method is Stevens' formula for the circle coverage probability.

If we place arcs of size a sequentially and uniformly on the circumference of a circle, Stevens' result yields the probability distribution of N, the random number of arcs at which complete coverage first occurs. This is because the probability of coverage by n arcs (where n is fixed) is $P(N\leqq n)$. L. Flatto and A. G. Konheim (1962) considered the problem asymptotically for a tending to zero (so that N increases stochastically without limit) and showed that

$$(4.16)\qquad E_a(N)=1-\sum_{k=1}^{[1/a]}(-1)^k\frac{(1-ka)^{k-1}}{(ka)^{k+1}}\sim a^{-1}\log a^{-1}\quad \text{as } a\searrow 0.$$

About ten years later, P. J. Cooke (1972) suggested the asymptotic form of the first two moments of N.

A. Dvoretsky (1956) considered coverage of the circle by an infinite number of random arcs of specified lengths but uniform independent placement. He showed that if the length of the ith arc is $l_i \in (0, 1)$, then the divergence of the sum $\sum_{i=1}^{\infty} l_i$ is necessary and sufficient for the arcs to cover almost all of the circle with probability one. He left open the corresponding problem of determining necessary and sufficient conditions on the sequence of lengths so as to cover all (not just "almost all") of the circle with probability one by exhibiting a sequence of lengths for which $\sum_{i=1}^{\infty} l_i$ diverged, but which did not cover all of the circle with probability one.

P. Billard (1965) gave a partial solution to Dvoretsky's problem by exhibiting a necessary condition and a (different) sufficient condition. M. Mandelbrot (1972) produced another partial solution and established for the first time that arcs of lengths $l_n = 1/(n+1)$ will cover all of the circle with probability one.

L. A. Shepp (1972) finally provided the complete answer to Dvoretsky's problem. He showed that if the length of the ith arc is $l_i \in (0, 1)$, then

$$\sum_{n=1}^{\infty} \frac{\exp\left(\sum_{i=1}^{n} l_i\right)}{n^2} = \infty \tag{4.17}$$

is a necessary and sufficient condition for these arcs, randomly placed, to cover all of the circle with probability one. Shepp also derived bounds on the probability of coverage of a circle by n arcs of size a, and used these bounds to study the asymptotic distribution of the number just needed to cover.

In this section we reconsider the problem of n arcs $X_1, \cdots, X_n$ of length a placed independently at random on the circumference of a circle and present the development by A. F. Siegel (1977). Assume that the circumference has length one and that the centers of the arcs (or equivalently their clockwise endpoints) are uniformly distributed over the circle. We define the *coverage* to be

$$C_{(n,a)} = \mu\left(\bigcup_{i=1}^{n} X_i\right) \tag{4.18}$$

and the *vacancy* to be

$$D_{(n,a)} = 1 - C_{(n,a)}, \tag{4.19}$$

where μ denotes Lebesgue measure on the circumference. Thus $C_{(n,a)}$ and $D_{(n,a)}$ are random variables representing the proportion of the circumference covered (respectively not covered) by the arcs $X_1, \cdots, X_n$. Siegel derives formulae for the moments of coverage (i.e., the moments of $C_{(n,a)}$), the distribution of coverage, and its limiting distribution as n tends to infinity.

Moments of coverage. The moments of coverage are found by deriving a recursive integral equation for the moments of $D_{(n,a)}$. Robbins' theorem (1944)

is the starting point, yielding a formula for the mth moment of $D_{(n,a)}$:

$$ED^m_{(n,a)} = \int_{K^m} P_a\left(z_1, \cdots, z_m \in \bigcup_{i=1}^{n} X_i^c\right) dz_1, \cdots, dz_m \tag{4.20}$$

where K denotes the circumference of our circle and the subscript a of P is the arc length. From the fact that $X_1, \cdots, X_n$ are independent and identically distributed sets, we have

$$ED^m_{(n,a)} = \int_{K^m} [P_a(z_1, \cdots, z_m \in X_1^c)]^n \, dz_1, \cdots, dz_m. \tag{4.21}$$

Using invariance of the integrand under permutations of the z_i to order them by their magnitudes, $0 \leqq z_{(1)} \leqq \cdots \leqq z_{(m)} < 1$, together with the fact that

$$\begin{aligned} &P_a(z_1, \cdots, z_m \in X_1^c) \\ &\quad = (z_{(2)} - z_{(1)} - a)_+ + \cdots + (z_{(m)} - z_{(m-1)} - a)_+ + (1 + z_{(1)} - z_{(m)} - a)_+ \end{aligned} \tag{4.22}$$

we can derive the recursive integral equation. For the case $a < 1/2$, this is

$$\begin{aligned} ED^{m+1}_{(n,a)} &= \binom{m+n}{n}^{-1} (1-a)^{m+n} \\ &\quad + m \sum_{k=1}^{n} \binom{n}{k} \int_a^{1-a} x^{m+k-1} (1-x-a)^{n-k} ED^m_{(k,a/x)} \, dx \\ &\quad + m \int_{1-a}^{1} x^{m+n-1} ED^m_{(n,a/x)} \, dx; \end{aligned} \tag{4.23}$$

whereas, if $a \geqq 1/2$, then the equation is simpler:

$$ED^{m+1}_{(n,a)} = \binom{m+n}{n}^{-1} (1-a)^{m+n} + m \int_a^1 x^{m+n-1} ED^m_{(n,a/x)} \, dx. \tag{4.24}$$

These may be solved. It can be shown by induction that the solution to (4.23) and (4.24) is

$$ED^m_{(n,a)} = \binom{m+n-1}{n}^{-1} \sum_{l=1}^{m} \binom{m}{l} \binom{n-1}{l-1} (1-la)_+^{m+n-1}, \tag{4.25}$$

where $(1-la)_+$ is equal to zero if $(1-la)$ is negative. The moments of coverage are then obtainable as

$$EC^m_{(n,a)} = E(1 - D_{(n,a)})^m = 1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} ED^k_{(n,a)}. \tag{4.26}$$

It may be instructive to write the first four moments of $D_{n,a}$ explicitly.

$$
\begin{aligned}
\mu_1 &= (1-a)^n,\\
\mu_2 &= \frac{2}{n+1}(1-a)^{n+1} + \frac{n-1}{n+1}(1-2a)_+^{n+1},\\
\mu_3 &= \frac{6}{(n+1)(n+2)}(1-a)^{n+2} + \frac{6(n-1)}{(n+1)(n+2)}(1-2a)_+^{n+2}\\
&\quad + \frac{(n-1)(n-2)}{(n+1)(n+2)}(1-3a)_+^{n+2},\\
\mu_4 &= \frac{24}{(n+1)(n+2)(n+3)}(1-a)^{n+3} + \frac{36(n-1)}{(n+1)(n+2)(n+3)}(1-2a)^{n+3}\\
&\quad + \frac{12(n-1)(n-2)}{(n+1)(n+2)(n+3)}(1-3a)_+^{n+3} + \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}(1-4a)_+^{n+3}.
\end{aligned}
\tag{4.27}
$$

Distribution of coverage. From the moments of coverage, the distribution of coverage $P(C_{(n,a)} \leqq t)$ is, in theory, completely determined because a distribution on a finite interval is uniquely characterized by its moments. By considering special cases, one may make an educated guess at the coverage distribution. Because we have the moments of coverage, this guess may be verified. By the use of this method, the cumulative distribution of coverage can be shown to be

$$
\begin{aligned}
F_{n,a}(t) &= P(C_{(n,a)} \leqq t)\\
&= \sum_{l=1}^{n} \sum_{k=0}^{l-1} (-1)^{l+k+1} \binom{n}{l} \binom{l-1}{k} \binom{n-1}{k} (1-t)^k (t-la)_+^{n-k-1}
\end{aligned}
\tag{4.28}
$$

where we use the convention that

$$
(t-la)_+^0 = \begin{cases} 1, & 1-la-t \geqq 0,\\ 0, & 1-la-t < 0. \end{cases}
\tag{4.29}
$$

The coverage, $C_{(n,a)}$, is a continuous random variable except for a point mass arising under one of two possible situations. First, if $na > 1$, then n random arcs of length a will have positive probability of covering the circle. In this case, $C_{(n,a)}$ has positive mass at 1. Second, if $na < 1$, then n arcs of length a will have positive probability of not overlapping. In this case, $C_{(n,a)}$ has positive mass at na.

In fact, we can decompose $C_{(n,a)}$ into a mixture of a continuous and a degenerate part:

$$
C_{(n,a)} = \begin{cases} A_{(n,a)}, & \text{probability } p,\\ B_{(n,a)}, & \phantom{\text{probability }} 1-p, \end{cases}
\tag{4.30}
$$

where

$$p=\begin{cases}\sum_{l=0}^{n}(-1)^l\binom{n}{l}(1-la)_+^{n-1}, & \text{if } na\geqq 1,\\ (1-na)^{n-1}, & \text{if } na<1,\end{cases} \tag{4.31}$$

$A_{(n,a)}$ is degenerate at $1-(1-na)_+$ because

$$A_{(n,a)}=\begin{cases}1, & na>1,\\ na, & na<1,\end{cases} \tag{4.32}$$

and $B_{(n,a)}$ has density

$$f_{n,a}(t)=\frac{n}{1-p}\sum_{l=1}^{n}\sum_{k=1}^{l}(-1)^{l+k}\binom{n-1}{l-1}\binom{n-1}{k}\binom{l-1}{k-1}(1-t)^{k-1}(t-la)_+^{n-k-1}, \tag{4.33}$$

and the functions $F_{n,a}(t)$ and $f_{n,a}(t)$ are piecewise polynomials.

Asymptotic distribution of coverage. If we hold arc length a constant and let n become large, the coverage probability tends to 1. If we condition on the event that the circle is not covered, we can study the conditional limiting distribution of coverage. This can be shown by the method of moments to be asymptotically related to the exponential distribution.

Using the decomposition (4.30), we see that the conditional coverage is

$$B_{(n,a)}=C_{(n,a)}|\{\text{circle not covered}\}. \tag{4.34}$$

From the moment formula (4.25) it can be shown that

$$\lim_{n\to\infty}E[n(1-B_{(n,a)})]^m=(1-a)^m m!. \tag{4.35}$$

Thus the limiting distribution of $n(1-B_{(n,a)})/(1-a)$ is exponential. This is to say that

$$\lim_{n\to\infty}P\left(\frac{n(1-B_{(n,a)})}{1-a}\leqq t\right)=1-e^t. \tag{4.36}$$

From this, it can be shown that

$$P\left(\frac{n(1-C_{(n,a)})}{1-a}>t\right)\sim n(1-a)^{n-1}e^{-t} \tag{4.37}$$

for each t, as n tends to infinity, in the sense that the limiting ratio of the two sides of (4.37) is 1.

Random caps on a sphere. The problem of random arcs covering a circle can be generalized to three dimensions, where the situation becomes one of covering the surface of a sphere with randomly placed circular caps. Consider a sphere of unit radius on which are placed N circular spherical caps subtending a half angle α at the center of the sphere (see Fig. 4.4). Assume the centers of these caps are

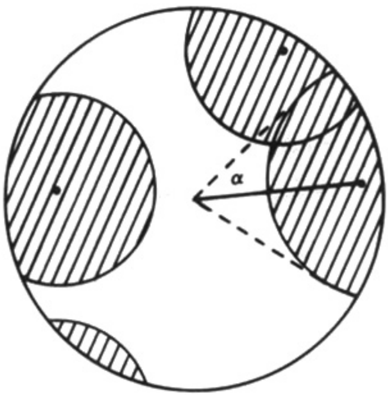


FIG. 4.4

independently and uniformly distributed over the surface of the sphere. We seek the probability $P(N)$ that the sphere is completely covered.

This problem arises in practice in the study of the attachment of antibodies to influenza virus particles. Antibodies are supposed to be cigar-shaped molecules which attach themselves at their ends to the virus particle. The influenza virus particle can be considered to be a sphere which infects a cell by coming in contact with it. If enough antibodies are attached to its surface, the virus cannot touch another cell; i.e., it is no longer infectious. In particular, since the cell which is susceptible to virus infection is much larger than the virus, its surface may be assumed flat (see Fig. 4.5). An attached antibody thus prevents a spherical cap S on the virus from touching and thus infecting the cell. The virus is no longer infectious when its surface is completely covered by the spherical caps centered at the attached antibodies.

The exact probability that a sphere is covered by N circular caps is not known. However, an asymptotic value for large N was derived by Moran and Fazekas de St. Groth (1962). Their method can be most easily demonstrated by applying it to the two dimensional problem of random arcs on the circle. Let Y be the

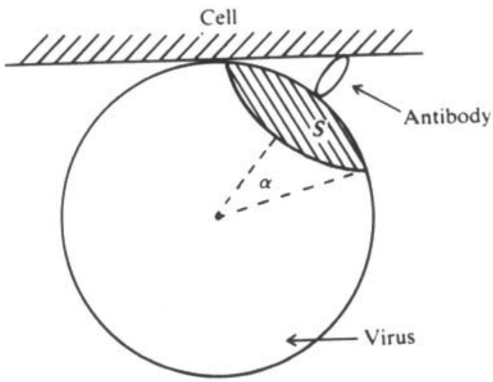


FIG. 4.5

measure of the set of points on the circumference not covered by the N arcs. Then the distribution of Y is clearly continuous on $[0, 1]$ with a concentration of probability $P(N)$ at $Y=0$.

We obtain the first two moments of Y in the following manner. Let the random variable $\chi_N(\xi)$ be the indicator function of the event that the point ξ is not covered by N random arcs; i.e.,

$$\chi_N(\xi)=\begin{cases}1 & \text{if } \xi \text{ is not covered by } N \text{ arcs,}\\ 0 & \text{otherwise.}\end{cases} \tag{4.38}$$

Then

$$Y=\int_0^1 \chi_n(\xi)\,d\xi \tag{4.39}$$

and

$$E[Y]=\int_0^1 E[\chi_n(\xi)]\,d\xi. \tag{4.40}$$

Clearly $E[\chi_1(\xi)]=(1-a)$. Since the arcs are independent

$$E[\chi_N(\xi)]=(E[\chi_1(\xi)])^N=(1-a)^N. \tag{4.41}$$

Thus we have

$$E[Y]=(1-a)^N. \tag{4.42}$$

Similarly

$$Y^2=\int_0^1 \chi_N(\xi)\,d\xi \int_0^1 \chi_N(\omega)\,d\omega \tag{4.43}$$

and

$$E[Y^2]=\int_0^1\int_0^1 E[\chi_N(\xi)\chi_N(\omega)]\,d\xi\,d\omega. \tag{4.44}$$

The expression $E[\chi_N(\xi)\chi_N(\omega)]$ is just the probability that both ξ and ω are not covered. If the arc length from ξ to ω is greater than a, then ξ and ω are not covered if and only if no clockwise endpoints of random arcs fall on the two sections of length a shown in Fig. 4.6a.

If the arc length from ξ to ω is less than a, say t, then ξ and ω are not covered if and only if no clockwise endpoints of random arcs fall on the section of length $t+a$ shown in Fig. 4.6b. Thus

$$\begin{aligned} E[\chi_N(\xi)\chi_N(\omega)] &= P[\xi, \omega \text{ not covered}]\\ &=\begin{cases}(1-2a)^N & \text{if } \|\xi-\omega\|\geqq a,\\ (1-(t+a))^N & \text{if } \|\xi-\omega\|=t<a.\end{cases}\end{aligned} \tag{4.45}$$

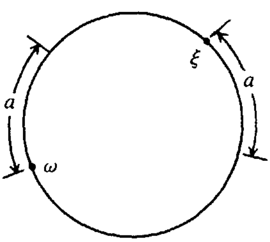


FIG. 4.6a

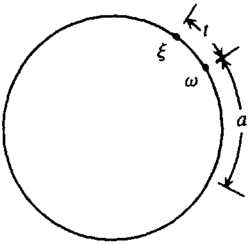


FIG. 4.6b

(Here $\|\xi-\omega\|$ denotes the positive arc length from ξ to ω.) Hence we have

$$E[Y^2]=\iint_{\substack{|\xi-\omega|\geqq x\\ 0\leqq \xi,\,\omega\leqq 1}}(1-2a)^N\,d\xi\,d\omega+\iint_{\substack{|\xi-\omega|<x\\ 0\leqq\xi,\,\omega\leqq 1}}[1-(|\xi-\omega|+a)]^N\,d\xi\,d\omega,$$

$$E[Y^2]=\int_{\|\xi-\omega\|>a}(1-2a)^N\,d\xi\,d\omega+\int_{\|\xi-\omega\|=t\leqq a}[1-(t+a)]^N\,d\xi\,d\omega$$

(4.46)
$$=\int_0^1\int_0^{\omega-a}(1-2a)^N\,d\xi\,d\omega+\int_0^1\int_{\omega+a}^1(1-2a)^N\,d\xi\,d\omega$$

$$+\int_0^1\int_{\omega-a}^{\omega}[1-(\omega-\xi+a)]^N\,d\xi\,d\omega+\int_0^1\int_{\omega}^{\omega+a}[1-(\xi-\omega+a)]^N\,d\xi\,d\omega$$

$$=\int_0^1(1-2a)^N(\omega-a)\,d\omega+\int_0^1(1-2a)^N(1-\omega-a)\,d\omega$$

$$+\frac{2}{N+1}\int_0^1[1-a]^{N+1}\,d\omega-\frac{2}{N+1}\int_0^1[1-2a]^{N+1}\,d\omega$$

$$=\frac{2}{N+1}[1-a]^{N+1}+\left[\frac{N-1}{N+1}\right][1-2a]^{N+1}.$$

For large N, the set of points which are not covered is with high probability a single interval if it is not empty. Furthermore, the length Y of this interval has a negative exponential distribution. (It may be thought of as the waiting time to the first endpoint of the series of overlapping arcs.) Let μ_1 denote the first moment about the origin of this negative exponential distribution. Then the second moment about the origin is $2\mu_1^2$ and we have

(4.47)
$$E(Y)=[1-P(N)]\mu_1,$$

(4.48)
$$E(Y^2)=2[1-P(N)]\mu_1^2.$$

Thus

(4.49)
$$1-P(N)=\frac{[E(Y)]^2}{\frac{1}{2}E(Y^2)}\sim N(1-a)^{N-1}.$$

Hence the probability that the circle is completely covered is

$$P(N) \sim 1 - N(1-a)^{N-1}. \tag{4.50}$$

Note that this is precisely what we would have arrived at using our previous result

$$P[\text{No gaps}] = \sum_{j=0}^{k} \binom{N}{j} (-1)^j (1-ja)^{N-1}.$$

For large N, the terms $(1-ja)^{N-1}$ for $j>1$ will be negligible compared with the term $(1-a)^{N-1}$ and we get

$$P[\text{No gaps}] \sim 1 - N(1-a)^{N-1} \tag{4.51}$$

which is the same as that obtained above.

Application of method of moments. We shall now apply this same method to obtain the probability that the surface of the sphere is completely covered by the spherical caps. Since the sphere has unit radius, its surface area is 4π. Each spherical cap of angular radius α will have an area of $2\pi(1-\cos\alpha)$. Thus the probability that any specified point on the sphere will be covered by a randomly placed cap is

$$\lambda = \frac{1}{2}(1-\cos\alpha) = \sin^2\frac{\alpha}{2}. \tag{4.52}$$

A ring of width $d\Phi$ and angular radius Φ (i.e., the ring subtends an angle 2Φ at the center of the sphere) on the surface of a sphere of unit radius has area $2\pi \sin\Phi \, d\Phi$ (see Fig. 4.7).

Thus the area of a circular cap of angular radius α is

$$\int_0^\alpha 2\pi \sin\Phi \, d\Phi = 2\pi(1-\cos\alpha). \tag{4.53}$$

Then the probability that N random caps do not cover some particular point P_1 is $(1-\lambda)^N$. Define the random variable $x_N(P_1)$ as above; that is, let $x_N(P_1)$ be the

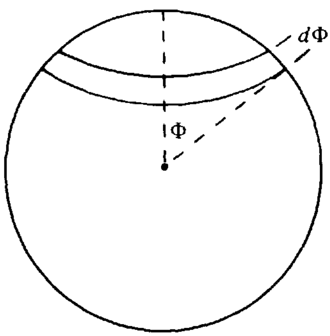


FIG. 4.7

indicator function of the event that the point P_1 is not covered by N random caps. Let Y be the proportion of the surface of the sphere not covered. Then $Y = \int x_N(P_1)\, dw_1$ where dw_1 is an element of surface area surrounding P_1. Then

$$E(Y) = \frac{1}{4\pi}\int E[x_N(P_1)]\, dw_1 = E[x_N(P_1)] = (1-\lambda)^N \tag{4.54}$$

and the second moment is

$$E(Y^2) = \frac{1}{16\pi^2}\int\int E[x_N(P_1)x_N(P_2)]\, dw_1\, dw_2. \tag{4.55}$$

Let P_1 and P_2 be separated by an angle Φ, and let $f(\Phi)$ denote the combined surface areas of the caps of angular radius α centered at P_1 and P_2. A random cap covers one of the two points if its center falls on the region of area $f(\Phi)$. Thus the probability that one random cap covers one of the points is $f(\Phi)/(4\pi)$. The probability that N random caps leave both P_1 and P_2 uncovered is

$$E[x_N(P_1)x_N(P_2)] = \left[1 - \frac{1}{4\pi}f(\Phi)\right]^N. \tag{4.56}$$

Since $E[x_N(P_1)x_N(P_2)]$ depends upon P_1 and P_2 only through Φ we can evaluate $E(Y^2)$ by using the following trick to integrate over $d\Phi$ instead of dw_1 and dw_2. The integral

$$E(Y^2) = \frac{1}{16\pi^2}\int\int E[x_N(P_1)x_N(P_2)]\, dw_1\, dw_2 \tag{4.57}$$

can be viewed as the expected value of a function $E[x_N(P_1)x_N(P_2)]$ of random points P_1 and P_2 which fall with a uniform distribution on the surface of the sphere. (Remember the expectation in $E[x_N(P_1)x_N(P_2)]$ refers to random caps, not the random points P_1 and P_2.) The two points P_1 and P_2 subtend an angle between Φ and $\Phi + d\Phi$ at the center of the sphere if and only if the second point falls in a ring of area $2\pi \sin\Phi\, d\Phi$ (see Fig. 4.8). This event happens with probability

$$\frac{2\pi \sin\Phi\, d\Phi}{4\pi} = \frac{1}{2}\sin\Phi\, d\Phi.$$

Thus we can substitute $\frac{1}{2}\sin\Phi\, d\Phi$ for $(d\omega_1/(4\pi))(d\omega_2/(4\pi))$ and we have

$$E(Y^2) = \frac{1}{2}\int_0^{\pi}\left[1 - \frac{1}{4\pi}f(\Phi)\right]^N \sin\Phi\, d\Phi. \tag{4.58}$$

When $\Phi \geqq 2\alpha$, the two circles around P_1 and P_2 do not overlap and $f(\Phi) = 4\pi(1-\cos\alpha)$. When $\Phi < 2\alpha$ we have (by spherical trigonometry)

$$1 - \frac{f(\Phi)}{4\pi} = \begin{cases} \left[1 - \dfrac{1}{\pi}\cos^{-1}\left(\dfrac{\tan\frac{1}{2}\Phi}{\tan\alpha}\right)\right]\cos\alpha + \dfrac{1}{\pi}\cos^{-1}\left(\dfrac{\sin\frac{1}{2}\Phi}{\sin\alpha}\right) & (0 \leqq \Phi \leqq 2\alpha) \\ \cos\alpha & (2\alpha \leqq \Phi \leqq \pi). \end{cases} \tag{4.59}$$

Thus $E(Y^2)$ can be evaluated by numerical integration.

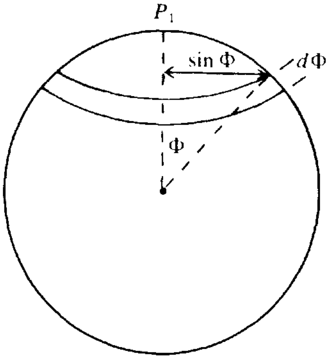


FIG. 4.8

We can therefore obtain the first two moments of the distribution of Y, the fraction of the total area not covered. This distribution will consist of a concentration of probability, $\Pr(N)$, at the origin together with continuous probability distribution on the range $(0 < Y < \infty)$ and can be written

$$\Pr[Y < y] = \Pr(N) + \int_0^y [1 - \Pr(N)] g(y)\, dy \tag{4.60}$$

where $g(y)$ is the density of Y given that $Y > 0$. In other words, for $Y > 0$ the density is just $[1 - \Pr(N)]g(y)$. Let μ_1 and μ_2 denote the first two moments of the density $g(y)$. Then

$$E(Y) = (1 - \Pr(N))\mu_1, \qquad E(Y^2) = (1 - \Pr(N))\mu_2$$

and

$$1 - \Pr(N) = \left(\frac{\mu_2}{\mu_1^2}\right)\frac{[E(Y)]^2}{E(Y^2)}. \tag{4.61}$$

Thus if we can find $\mu_2\mu_1^{-2}$, $\Pr(N)$ can be obtained from the above expression.

To obtain μ_1 and μ_2 we note that for large N the size of any uncovered area on the sphere's surface will be small compared with the radius of the sphere and with the radius of the random caps. Thus the curvature of the surface of the sphere can be neglected, the region can be considered to be planar, and the boundary of the region may be taken to be made up of straight lines. Furthermore since the caps are uniformly distributed on the surface of the sphere, the straight lines are random lines of a Poisson field in the plane. Thus the area of any single (i.e. connected) uncovered region has approximately the same distribution as the area, A, of a polygon formed by random lines of a Poisson field in the plane. We have already obtained the values of $E(A)$ and $E(A^2)$ during our discussion of random lines in the plane, namely

$$E(A) = \pi/\tau^2, \qquad E(A^2) = \frac{1}{2}\pi^4/\tau^4$$

where τ is the intensity of the random lines in the Poisson field.

In the example with the random arcs on the circle we saw that for large N the probability of more than one disjoint uncovered area was negligible. If we make the similar assumption that the probability of more than one disjoint uncovered areas on the sphere is negligible compared with the probability of one such area, we get $\mu_1 = E(A)$ and $\mu_2 = E(A^2)$. Then

$$\frac{\mu_2}{\mu_1^2} = \frac{1}{2}\pi^2$$

and

$$\text{(4.62)} \qquad \Pr(N) = 1 - \frac{1}{2}\pi^2[E(Y)]^2/E(Y^2).$$

This formula is not satisfactory for low values of N as it yields negative values of $\Pr(N)$. An alternative hypothesis is that the number of uncovered areas, M, is distributed in a Poisson distribution with mean ν. This cannot be true when the random caps are large for in the cases where $\alpha \geqq 90°$ there cannot be more than one region uncovered. For smaller values of α, however, this assumption seems plausible. We also assume that the areas, Y_i, of the individual regions are distributed independently. If the total area is Y we have $Y = Y_1 + \cdots + Y_M$ and using the moments of the Poisson distribution we get

$$E(Y) = \nu E(Y_i)$$

and

$$E(Y^2) = \nu E(Y_i^2) + \nu^2 (EY_i)^2.$$

If we now put $E(Y_i^2)[E(Y_i)]^{-2} = \frac{1}{2}\pi^2$ we have

$$\text{(4.63)} \qquad \nu = \frac{1}{2}\pi^2\{E(Y^2)[E(Y)]^{-2} - 1\}^{-1}$$

and the probability that the random caps do not cover the sphere is

$$\text{(4.64)} \qquad \begin{aligned} 1 - \Pr(N) &= 1 - \exp(-\nu) \\ &= 1 - \exp\left(-\frac{1}{2}\pi^2\{E(Y^2)[E(Y)]^{-2} - 1\}^{-1}\right). \end{aligned}$$

Moran and Fazekas de St. Groth tested the accuracy of the above formula with an experiment using ping-pong balls. One hundred circular holes were punched in an aluminum sheet. The diameter of these holes was chosen so that the part of a ping-pong ball which may protrude through each hole is a spherical cap of angular radius $\alpha = 53.43°$ (i.e. subtending a half angle of 53.43° at the center of the ping-pong ball). This value of α was chosen because for the influenza virus the radius of the sphere is 40 mμ while the antibodies are of length 27 mμ so that the shielded area subtends a half angle of 53.43°. One hundred ping-pong balls were placed in the holes and were held firmly against the holes by a foam rubber pad. The spherical caps visible from the other side of

the aluminum sheet were sprayed with paint, thus simulating the dropping of a random cap of paint on each ping-pong ball. After drying, the ping-pong balls were removed, rotated randomly and replaced. Then another random cap of paint was added to each ball. The following table (Table 2) shows the number of balls not completely covered with paint after N sprayings in each of three times the experiment was carried out.

TABLE 2

		Number of balls not covered		
N	$1-\Pr(N)$	1st expt.	2nd expt.	3rd expt.
10	1.0000	100	100	100
15	.9846	95	96	97
20	.8009	76	82	80
25	.4916	53	62	52
30	.2502	28	32	24
35	.1143	12	19	11
40	.0489	6	8	3

An exact expression for $\Pr(N)$ is presently not known except for several special values of α. These special cases include the following:

1) $\Pr(N)=0$ if $N \sin^2(\alpha/2) \leqq 1$. This is obvious when we remember that $\sin^2(\alpha/2)$ is the fraction of the sphere's total surface covered by one cap.

2) $\Pr(2)=\cos^2\alpha$ if $\alpha > 90°$. The portion of the sphere not covered by one cap is itself a spherical cap of angular radius $2\pi-2\alpha$, which will be covered by a second random cap if and only if the center of the second random cap falls at great circle distance $2\pi-2\alpha$ or more from the center of the first random cap (see Fig. 4.9).

Thus the probability that the sphere is not completely covered is just the probability that the center of the second random cap falls in a cap shaped area (C

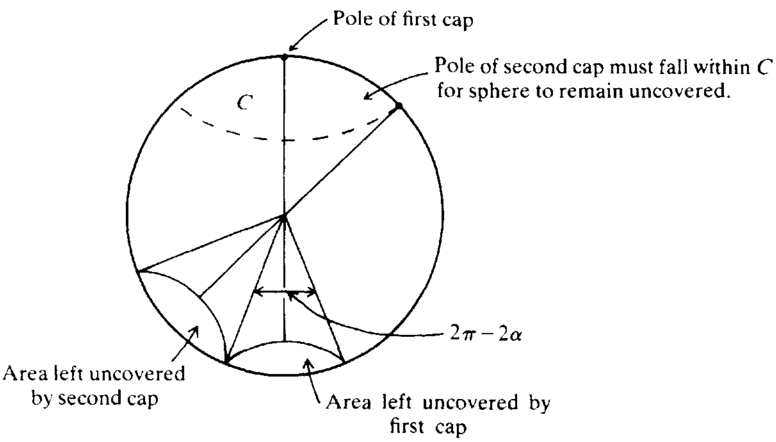


FIG. 4.9

in Fig. 4.9) of angular radius $2\pi - 2\alpha$ centered at the center of the first cap. This probability is just $\sin^2 \frac{1}{2}(2\pi - 2\alpha)$ or $\sin^2 \alpha$. Hence the probability that the sphere is completely covered is $\Pr(N) = 1 - \sin^2 \alpha = \cos^2 \alpha$.

3) $\Pr(N) = 1 - 2^N(N^2 - N + 2)$ when $\alpha = 90°$. When $\alpha = 90°$ each random cap is a hemisphere. The surface of the sphere is cut up into $N^2 - N + 2$ pieces by the N great circles which are the circumferences of the N random hemispheres. (This can be seen by the same argument as was used in the derivation of $E(N)$ for random lines in the plane, namely: One great circle splits the surface of the sphere into two pieces, and the Nth great circle intersects each of the previous $N-1$ great circles twice, thereby cutting $2(N-1)$ of the previous pieces in half, or adding $2(N-1)$ to the total number of pieces. Thus if M_N denotes the number of pieces into which N great circles cut up the surface of the sphere, we have $M_N = M_{N-1} + 2(N-1)$ or $M_N = 2\sum_{j=2}^{N}(j-1) + 2 = N^2 - N + 2$.) To each great circle there are two equally probable ways the cap could have landed and still have had the same great circle as a circumference. For example, the center of the cap could be either the North pole or the South pole with equal probability. Thus, given the N great circles and the pieces into which they cut the surface of the sphere, each piece has probability $\frac{1}{2}$ of not being covered by a certain random cap and probability 2^{-N} of not being covered by any of them. Since there are $N^2 - N + 2$ pieces, the probability is $2^{-N}(N^2 - N + 2)$ that any piece is not covered by any of the N random caps. Thus the probability that the whole surface of the sphere is covered is

$$\Pr(N) = 1 - 2^{-N}(N^2 - N + 2)$$

when $\alpha = 90°$.

This result is a special case of a more general result in geometric probability due to J. G. Wendel (1962). He showed that if N points are scattered at random on the surface of the unit sphere in n-space, the probability that all the points lie on some hemisphere is $2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k}$.

In addition to these few special cases for which an exact expression for $\Pr(N)$ can be developed, we can obtain upper and lower bounds for $\Pr(N)$ which can be combined to yield the exact asymptotic behavior of $\Pr(N)$ as N goes to infinity. These bounds were first derived by E. N. Gilbert (1965).

An upper bound for $\Pr(N)$ is easy to obtain. Any particular point, say the North pole, has probability $(1-\lambda)^N$ of being left uncovered by all N caps, where $\lambda = \sin^2 \frac{1}{2}\alpha$ is the fraction of the sphere's surface area which is covered by a single cap. Thus the probability that the North pole is covered by at least one cap is $1 - (1-\lambda)^N$. Since the sphere is not covered unless the North pole is covered, we have

$$\Pr(N) \leqq 1 - (1-\lambda)^N. \tag{4.65}$$

To obtain a lower bound for $\Pr(N)$ let us define a crossing as being a point of intersection of the circular boundaries of two overlapping caps. See Fig. 4.10a. If there is an uncovered crossing, some area of the sphere outside of the two

overlapping caps must also be uncovered. Conversely, if at least two caps overlap, then the boundary of any uncovered area on the sphere must contain an uncovered crossing. Thus we find that the sphere is completely covered if and only if there are at least two caps which overlap and every crossing is covered.

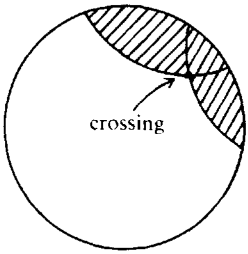


FIG. 4.10a

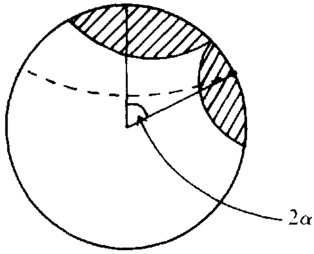


FIG. 4.10b

A cap intersects another cap if its center falls within a great circle distance of 2α from the center of the other cap; that is, if its center falls in a large cap of angular radius 2α centered at the center of the other cap (see Fig. 4.10b). This event has probability equal to the fraction of the sphere's area which is covered by the large cap of angular radius 2α, namely, $\sin^2\alpha$. Clearly for every pair of overlapping caps there are two crossings. Thus the expected number of crossings between two particular caps is $2\sin^2\alpha$. If we drop N caps, there are $\binom{N}{2}$ or $N(N-1)/2$ distinct pairs of caps and the expected number of crossings between all caps is

$$(4.66)\qquad \begin{aligned} N(N-1)\sin^2\alpha &= 4N(N-1)\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} \\ &= 4N(N-1)\lambda(1-\lambda). \end{aligned}$$

The probability that a crossing will not be covered by the remaining $N-2$ caps is $(1-\lambda)^{N-2}$. Thus the expected number of uncovered crossings between N caps on the sphere is $4N(N-1)\lambda(1-\lambda)^{N-1}$. If we denote $w(k)$ as the probability that the N caps leave exactly k crossings uncovered, we have

$$(4.67)\qquad \sum_{k=1}^{\infty} kw(k) = 4N(N-1)\lambda(1-\lambda)^{N-1}.$$

Let $U(N)$ denote the conditional expectation of the number of uncovered crossings given that not all crossings are covered. Thus

$$(4.68)\qquad U(N) = \sum_{k=1}^{\infty} kw(k)/(1-w(o)).$$

Therefore we have

$$(4.69)\qquad 1-w(o) = 4N(N-1)\lambda(1-\lambda)^{N-1}/U(N).$$

But $w(o)$ is the probability that all crossings are covered, which is the same as $\Pr(N)$. Also, $U(N) \geqq 3$ since an uncovered crossing occurs only if there is an uncovered area on the sphere, which requires at least three crossings in its boundary. Thus we have

$$\Pr(N) \geqq 1 - \frac{4}{3}N(N-1)\lambda(1-\lambda)^{N-1}. \tag{4.70}$$

For small N this is not very good, in fact it can be negative. On the other hand, this lower bound can be combined with our previously obtained upper bound to get

$$\log(1-\lambda) \leqq \frac{1}{N}\log(1-\Pr(N)) \leqq \frac{N-1}{N}\log(1-\lambda) + \frac{1}{N}\log\left[\frac{4}{3}N(N-1)\lambda\right].$$

Since the limit of the second summand on the right is zero, we have the asymptotic behavior of $\log(1-\Pr(N))$, i.e.,

$$\begin{aligned} \lim_{N\to\infty} N^{-1}\log(1-\Pr(N)) &= \log(1-\lambda) \\ &= \log\left(1-\sin^2\frac{\alpha}{2}\right). \end{aligned} \tag{4.71}$$

CHAPTER 5

Crofton's Theorem and Sylvester's Problem in Two and Three Dimensions

Crofton's theorem on mean values. It is to M. W. Crofton, an English mathematician whose works appear in the latter half of the 19th century, that we are indebted for approaches by which results in geometrical probability that normally would require difficult and sometimes intractable integration are achieved by ingenious artifices. His methods permit the evaluation of some definite integrals without directly performing the integrations. In previous chapters we have referred to some of his work. We now discuss some other situations, develop Crofton's very useful mean value theorem, and apply it in some classical problems such as Sylvester's four point problem—namely, find the probability that four points taken at random within a given boundary shall form a re-entrant quadrilateral. An especially pertinent article by Crofton is the one he prepared for the 9th edition of the Encyclopaedia Britannica (1885), particularly the section entitled "On Local Probability". We now reproduce Crofton's results.

Let us now consider a few specific questions. Assume that a specified space S is included within a given space A. Let a point P be taken at random on A and let p be the probability that it falls on S or we may write

$$p = S/A.$$

Now assume the space S is variable and that it has n equally probable values $S_1, S_2, S_3, \cdots, S_n$. Then the chance that any one S_i, say S_1, is taken and that then P falls on S_1 is

$$p_1 = \left(\frac{1}{n}\right)(S_1/A).$$

Let $p = p_1 + p_2 + \cdots + p_n$ and so

$$p = \left(\frac{S_1 + S_2 + \cdots + S_n}{n}\right)\left(\frac{1}{A}\right) = M(S)/(A) \tag{5.1}$$

where $M(S)$ is the mean value of S.

The chance of two points falling on S given that they fall at random on A is similarly

$$p = M(S^2)/A^2. \tag{5.2}$$

where $M(S^2)$ is the mean value of the random variable S^2. Note in these two illustrations that if the probability is known, the mean value follows, and vice versa.

Let us explore this in connection with the following problem. Consider a line of length l and the distance XY between two points falling at random on l—here l is A and XY is S. We wish to determine $M(XY)^n$, namely the nth moment of the random distance XY. To do this, consider the probability that n (additional) points taken at random on l fall in XY. Then

$$p=\frac{2}{(n+1)(n+2)}, \tag{5.3}$$

for the probability that X is one of the extreme points out of $(n+2)$ points is $2/(n+2)$, and, if it is, the probability that Y is the other extreme point is $1/(n+1)$. However, we also have

$$p=\frac{M(XY)^n}{l^n} \tag{5.4}$$

and so

$$M(XY)^n=\frac{2l^n}{(n+1)(n+2)}. \tag{5.5}$$

Let us consider another illustration by which the relationship between mean values and probabilities can be exploited. A line of length l is divided into n segments by $(n-1)$ points taken at random on the line. We wish to find the mean value of the product of the lengths of these segments. Let $a, b, c, \cdots$ be the n segments in one particular case and let n new points be taken at random over the line. The probability that one falls on each segment is

$$n!\frac{a}{l}\frac{b}{l}\frac{c}{l}\cdots=n!\frac{abc\cdots}{l^n}.$$

Now we seek the same probability no matter how the line is divided by the $(n-1)$ random points. This is

$$\frac{n!}{l^n}M(abc\cdots). \tag{5.6}$$

Now the number of different arrangements in which all $(2n-1)$ points may occur on the line is $(2n-1)!$. Out of these arrangements, the number in which a point of the first series falls between every two of the second series is from permutation theory equal to $(n!)(n-1)!$. Thus

$$\frac{n!}{l^n}M(abc\cdots)=\frac{n!(n-1)!}{(2n-1)!}$$

and

$$M(abc\cdots)=\frac{(n-1)!}{(2n-1)!}l^n. \tag{5.7}$$

We shall now derive an important formula known as Crofton's formula or Crofton's theorem on fixed points. Let n points $\xi_1, \xi_2, \cdots, \xi_n$ be randomly distributed on a domain S and let H be some event which depends on the positions of the n points. Let S' denote a domain slightly smaller than S and contained in S. Denote by ΔS the part of S not in S'. If we take ΔS small enough that we may neglect the probability of two or more of the ξ_i falling in ΔS, we get (where P now stands for probability)

$$\begin{aligned} P[H] &\cong P[H \mid \xi_1, \xi_2, \cdots, \xi_n \in S'] P[\xi_1, \xi_2, \cdots, \xi_n \in S'] \\ &\quad + \sum_{j=1}^{n} P[H \mid \xi_j \in \Delta S, \xi_i \in S' \text{ for all } i \neq j] P[\xi_j \in \Delta S] P[\xi_i \in S' \text{ for all } i \neq j]. \end{aligned} \tag{5.8}$$

Let s, s' and Δs denote the measures (e.g., area, volume, etc.) of S, S' and ΔS respectively. Since the points are random in S

$$P[\xi_j \in \Delta S] = \frac{\Delta s}{s} \tag{5.9}$$

and

$$P[\xi_1, \xi_2, \cdots, \xi_n \in S'] = \left[\frac{s'}{s}\right]^n. \tag{5.10}$$

Then we get

$$\begin{aligned} P[H]s^n &\cong P[H \mid \xi_1, \xi_2, \cdots, \xi_n \in S'] s'^n \\ &\quad + nP[H \mid \xi_1 \in \Delta S, \xi_2, \cdots, \xi_n \in S'] s'^{n-1} \Delta S. \end{aligned} \tag{5.11}$$

Since the domain ΔS is taken to be small enough that we can throw away the higher order terms of ΔS, we have

$$s^n = (s' + \Delta s)^n \cong s'^n + ns'^{n-1} \Delta s. \tag{5.12}$$

(These approximations will become exact when we let Δs become infinitesimal.) Thus we have

$$\begin{aligned} P[H](s'^n + ns'^{n-1} \Delta s) &\cong P[H \mid \xi_1, \xi_2, \cdots, \xi_n \in S'] s'^n \\ &\quad + nP[H \mid \xi_1 \in \Delta S, \xi_2, \cdots, \xi_n \in S'] s'^{n-1} \Delta s \end{aligned} \tag{5.13a}$$

or

$$\begin{aligned} & s'(P[H] - P[H \mid \xi_1, \xi_2, \cdots, \xi_n \in S']) \\ &\qquad \cong n\, \Delta s (P[H \mid \xi_1 \in \Delta S, \xi_2, \cdots, \xi_n \in S'] - P[H]). \end{aligned} \tag{5.13b}$$

We now let Δs become infinitesimal. Then S' is approximately S so that s' approaches s, the difference $P[H] - P[H \mid \xi_1, \xi_2, \cdots, \xi_n \in S']$ is a small increment $\delta P[H]$, ΔS becomes δs, and $P[H \mid \xi_1 \in \Delta S, \xi_2, \cdots, \xi_n \in S']$ becomes the probability $P[H \mid \xi_1 \in \delta S]$ that H occurs when one of the random points is on the boundary δS of S. Thus we arrive at Crofton's formula

$$\delta P[H] = n(P[H \mid \xi_1 \in \delta S] - P[H]) s^{-1}\, \delta s. \tag{5.14}$$

As a simple example of an application of Crofton's formula suppose ξ_1 and ξ_2 are random points on an interval of length L. Let $X=|\xi_1-\xi_2|$ and let $F(x, L)$ denote the cumulative distribution function of X, that is $F(x, L)=P[X<x]$. By Crofton's formula $\delta F(x, L)=2(P[X<x \mid \xi_1 \text{ at end of interval}]-F(x, L))L^{-1}\,\delta L$ since ξ_2 is uniformly distributed on the interval

$$P[X<x \mid \xi_1 \text{ at end of interval}]=x/L \tag{5.15}$$

and (we assume $dF(x, L)/dL$ exists)

$$\frac{dF(x, L)}{dL}=\frac{2}{L}(x/L-F(x, L)) \tag{5.16}$$

or

$$L^2\frac{dF(x, L)}{dL}+2LF(x, L)=2x,$$

$$\frac{d}{dL}(L^2F(x, L))=2x,$$

$$L^2F(x, L)=2xL+\text{constant}. \tag{5.17}$$

For $L=x$ we want $F(x, L)=1$. Thus the constant is $-x^2$ and

$$F(x, L)=\frac{1}{L^2}(2Lx-x^2). \tag{5.18}$$

There is a form of Crofton's formula which applies to mean values. If X is a random variable which depends on the positions of n random points $\xi_1, \xi_2, \cdots, \xi_n$ in S and if S' and ΔS are as before, we have for small ΔS

$$\begin{aligned} E(X) &\cong E(X \mid \xi_1, \xi_2, \cdots, \xi_n \in S')P[\xi_1, \xi_2, \cdots, \xi_n \in S'] \\ &\quad+\sum_{j=1}^{n} E(X \mid \xi_j \in \Delta S, \xi_i \in S' \text{ for all } i \neq j)P[\xi_j \in \Delta S]\,P[\xi_i \in S' \text{ for all } i \neq j]; \end{aligned} \tag{5.19}$$

if, as before, s, s' and Δs denote the measures of S, S' and ΔS respectively, we have

$$\begin{aligned} E(X)s^n &\cong E(X \mid \xi_1, \xi_2, \cdots, \xi_n \in S')s'^n \\ &\quad+nE(X \mid \xi_1 \in \Delta S, \xi_2, \cdots, \xi_n \in S')s'^{n-1}\,\Delta s \end{aligned} \tag{5.20}$$

and by the same arguments as before we get

$$\delta E(X)=n[E(X \mid \xi_1 \in \delta S)-E(X)]s^{-1}\,\delta s. \tag{5.21}$$

A more general form of Crofton's theorem has been provided in recent years by Ruben and Reed (1973). This extension concerns problems in which points are chosen at random in each of a number of domains. Assume for $j=1, \cdots, k$; D_j is a domain in n_j-dimensional Euclidean space with content V_j, and that N_j points are chosen at random from D_j. Now let μ be the expected value of some

function of $\sum_{j=1}^{k} N_j$ points that depends only on the intrinsic properties of the points and their relative positions and not on the domains D_j or the positions of the points relative to them. Then the extended Crofton theorem states that

$$d\mu = \sum_{j=1}^{k} N_j(\mu_j^* - \mu)\frac{dV_j}{V_j} \tag{5.22}$$

where $d\mu$ is the increment in μ obtained by increasing such D_j by an infinitesimal increment dV_j, and μ_j^* is the expected value of the function where one point is chosen at random in the boundary in D_j, $N_j - 1$ points are chosen at random in D_j and N_i points are chosen at random in D_i for $i \neq j$, $i = 1, 2, \cdots, k$.

As in the original Crofton theorem a direct corollary is that if P is the probability that $\sum_{j=1}^{k} N_j$ points satisfy some intrinsic property then

$$dP = \sum_{j=1}^{k} N_j(P_j^* - P)\frac{dV_j}{V_j}. \tag{5.23}$$

This result follows from the extended theorem by considering an indicator function with values 1 or 0 if the property is or is not satisfied.

The extended theorem allows repeated application, and this enables the total degrees of freedom of points selected at random in, for example, an n-simplex or polyhedron, in n-space, to be reduced by $n+1$. The expected value of a function of points can be decomposed into a linear combination of expectations, the degree of expectation in each of which has been reduced by $n+1$. In the original Crofton theorem, the degrees of the freedom of the points is reduced by one, and thus the degree of integration involved in finding the expectation is reduced by one.

Sylvester's four point problem. Many complex problems are solved through the use of Crofton's formula. One example is the famous Sylvester four point problem, which we shall now consider. In the late nineteenth century the English mathematician J. J. Sylvester posed the problem of finding the probability that four points taken at random inside a convex domain D form a re-entrant quadrilateral, that is, one of the points lies inside the triangle formed by the other three. A re-entrant quadrilateral can be formed from four points in four different ways, according to which of the four points occurs inside the triangle formed by the other three. If the domain D has area A_D and the mean area of a triangle formed by three random points X, Y and Z in the domain is $E(\text{Area } XYZ)$ then the probability Pr [RQ] of a re-entrant quadrilateral is

$$\Pr[\text{RQ}] = 4E(\text{Area } XYZ)/A_D; \tag{5.24}$$

clearly we can reduce or expand the scale of the domain D and the ratio $E(\text{Area } XYZ)/A_D$ remains unaffected. (In this section, we use Pr to denote the probability.) Therefore the probability of four random points forming a re-entrant quadrilateral in D is the same as if the four points were random in a

slightly smaller domain with the same shape. Thus we have $\delta \Pr[\mathrm{RQ}]=0$; in Crofton's formula

$$\delta \Pr[\mathrm{RQ}] = 4(\Pr[\mathrm{RQ} \mid \text{one point on boundary}] - \Pr[\mathrm{RQ}])A_D^{-1}\,\delta A_D \tag{5.25}$$

which implies that

$$\Pr[\mathrm{RQ}] = \Pr[\mathrm{RQ} \mid \text{one point on boundary}]. \tag{5.26}$$

Let O denote the random point on the boundary δD of D. Given that $O \in \delta D$, a re-entrant quadrilateral can be formed from three random points in D and $O \in \delta D$ in three ways, according to which of the three random points in D occurs inside of the triangle formed by O with the other two random points. Thus if $E(\text{Area } OXY \mid O \in \delta D)$ denotes the mean area of a triangle formed by X and Y random in D and O random on δD, we have

$$\Pr[\mathrm{RQ}] = \Pr[\mathrm{RQ} \mid O \in \delta D] = 3E(\text{Area } OXY \mid O \in \delta D)/A_D. \tag{5.27}$$

We shall now evaluate $E(\text{Area } OXY \mid O \in \delta D)$ and $\Pr[\mathrm{RQ}]$ for several types of convex domains D.

Let us first suppose D is a triangle, ABC. Figure 5.1 shows the triangle ABC together with the random point O on the boundary. Let S_1 denote the part of triangle to the left of the line AO and S_2 the part to the right of AO. We shall use the subscript O in expectations such as $E_O(\text{Area } OXY)$ to indicate that the expectation is over the distributions of only X and Y and that O is treated as a fixed point on the boundary. Thus for any fixed point O on the boundary the expected area of the triangle formed by O and two random points X and Y in ABC is

$$\begin{aligned} E_O(\text{Area } OXY) = {} & E_O(\text{Area } OXY \mid X, Y \in S_1)\Pr[X \in S_1]\Pr[Y \in S_1] \\ & + E_O(\text{Area } OXY \mid X, Y \in S_2)\Pr[X \in S_2]\Pr[Y \in S_2] \\ & + E_O(\text{Area } OXY \mid X \in S_1, Y \in S_2)\Pr[X \in S_1]\Pr[Y \in S_2] \\ & + E_O(\text{Area } OXY \mid X \in S_2, Y \in S_1)\Pr[X \in S_2]\Pr[Y \in S_1]. \end{aligned} \tag{5.28}$$

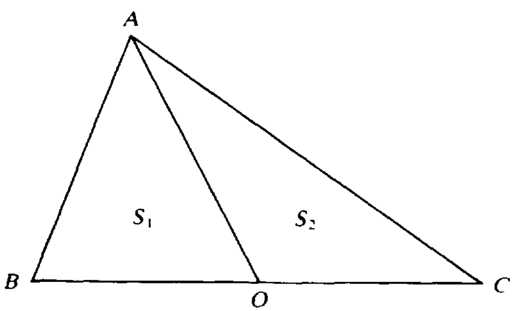


FIG. 5.1

If we denote the areas of S_1, S_2 and ABC by s_1, s_2 and A_D respectively, we get

$$
\begin{aligned}
E_O(\text{Area } OXY) = {} & E_O(\text{Area } OXY \mid X,\ Y \in S_1) s_1^2/A_D^2 \\
(5.29) \qquad & + E_O(\text{Area } OXY \mid X,\ Y \in S_2) s_2^2/A_D^2 \\
& + 2E_O(\text{Area } OXY \mid X \in S_1,\ Y \in S_2) s_1 s_2/A_D^2.
\end{aligned}
$$

In order to calculate $E_O(\text{Area } OXY)$ we must evaluate the conditional expectations on the right hand side of the equation. We shall evaluate $E_O(\text{Area } OXY \mid X \in S_1,\ Y \in S_2)$ first. Clearly the expected value of the distance from a random point $X \in S_1$ to a line not passing through S_1 is the distance from the line to the center of gravity G_1 of S_1. Then for a fixed point $Y_0 \in S_2$ the expected area of the triangle OXY_0 over all positions of $X \in S_1$ is

$$
\begin{aligned}
E_O(\text{Area } OXY \mid X \in S_1,\ Y = Y_0) &= \frac{1}{2}\|OY_0\| E_O(\|OY_0 \text{ to } X\| \mid Y = Y_0) \\
(5.30) \qquad &= \frac{1}{2}\|OY_0\|\,\|OY_0 \text{ to } G_1\| \\
&= \text{Area } OY_0G_1
\end{aligned}
$$

where $\|OY_0\|$ denotes the length of the line segment OY_0 and $\|OY_0 \text{ to } G_1\|$ denotes the perpendicular distance from G_1 to the line OY_0. The area of the triangle OY_0G_1 can also be written

$$
(5.31) \qquad E_O(\text{Area } OXY \mid X \in S_1,\ Y = Y_0) = \frac{1}{2}\|OG_1\|\,\|OG_1 \text{ to } Y_0\|.
$$

The mean value $E_O(\text{Area } OXY \mid X \in S_1,\ Y \in S_2)$ is the expectation of $E_O(\text{Area } OXY \mid X \in S_1,\ Y = Y_0)$ over all $Y_0 \in S_2$. Thus

$$
\begin{aligned}
E_O(\text{Area } OXY \mid X \in S_1,\ Y \in S_2) &= E_O(E_O(\text{Area } OXY \mid X \in S_1,\ Y)) \\
&= \frac{1}{2}\|OG_1\| E_O(\|OG_1 \text{ to } Y\|) \\
(5.32) \qquad & \\
&= \frac{1}{2}\|OG_1\|\,\|OG_1 \text{ to } G_2\| \\
&= \text{Area } OG_1G_2
\end{aligned}
$$

where G_2 is the center of gravity of S_2.

We shall now evaluate the area of the triangle OG_1G_2. (See Fig. 5.2.) We shall show that

$$
(5.33) \qquad \text{Area } OG_1G_2 = \frac{1}{9} \text{Area } ABC.
$$

We shall also show that

$$
(5.34) \qquad \text{Area } AG_1G_2 = \frac{2}{9} \text{Area } ABC
$$

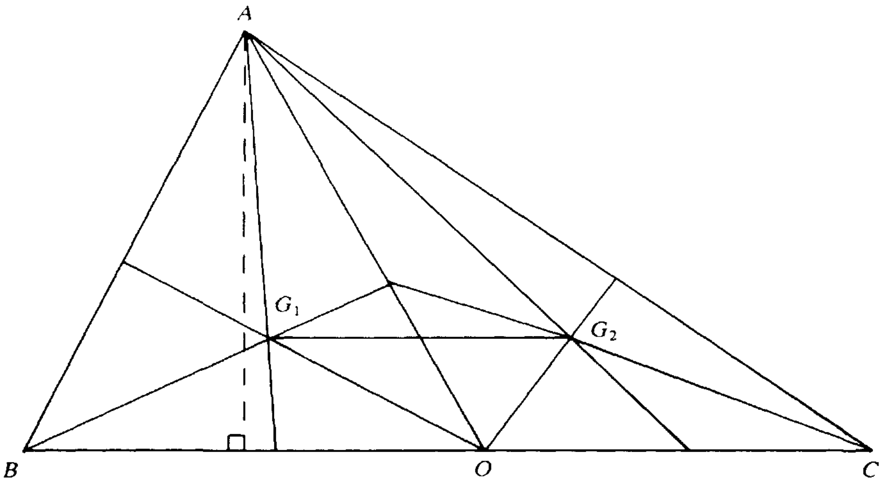


FIG. 5.2

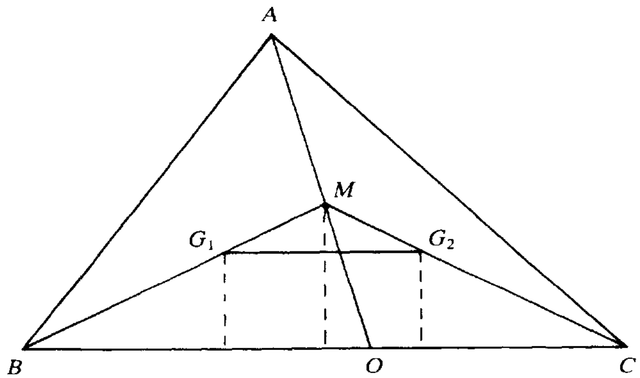


FIG. 5.3

a fact which we will need later on. We use the fact that the three medians of a triangle are concurrent at the center of gravity, which is two thirds of the way along each median from the vertex to the opposite side. Hence G_1 is the point of intersection of the medians of ABO and G_2, the point of intersection of the medians to AOC. Since G_1 and G_2 are the same fraction, 2/3 of the way from A to the line BC, the line G_1G_2 must be parallel to BC. Furthermore the line G_1G_2 extended intersects any line perpendicular to the base BC (for example the broken line in Fig. 5.2) two thirds of the way from A to the line BC. Thus we have

$$\text{Altitude } OG_1G_2 = \frac{1}{3} \text{ Altitude } ABC \tag{5.35}$$

and

$$\text{Altitude } AG_1G_2 = \frac{2}{3} \text{ Altitude } ABC. \tag{5.36}$$

Since G_1 occurs two thirds of the way along the median BM (see Fig. 5.3), the projection of BG_1 on the base BC is two thirds of the projection of BM on BC. Similarly the projection of CG_2 on BC is two thirds the length of the projection on BC of the median CM. Hence the projection of G_1G_2, which is BC minus the projections of BG_1 and CG_2, is one third the length of BC. Since G_1G_2 is parallel to BC, we have

$$\|G_1G_2\| = \frac{1}{3}\|BC\|. \tag{5.37}$$

Combining this result with our previous results for the altitudes of OG_1G_2 and AG_1G_2 we have

$$\text{Area } OG_1G_2 = \frac{1}{9} \text{ Area } ABC, \qquad \text{Area } AG_1G_2 = \frac{2}{9} \text{ Area } ABC. \tag{5.38}$$

Thus we have

$$E_O(\text{Area } OXY \mid X \in S_1,\ Y \in S_2) = \text{Area } OG_1G_2 = \frac{1}{9} A_D. \tag{5.39}$$

We shall now evaluate $E_O(\text{Area } OXY \mid X, Y \in S_1)$. We must first prove the following fact. For any two triangles OLR and $O'L'R'$ with random points X and Y in OLR and X' and Y' in $O'L'R'$, we have

$$\frac{E(\text{Area } OXY)}{\text{Area } OLR} = \frac{E(\text{Area } O'X'Y')}{\text{Area } O'L'R'}. \tag{5.40}$$

This fact is not immediately apparent but can be seen in the following way. The triangle OLR can be transformed into $O'L'R'$ by projecting it obliquely onto another plane and changing the scale. The projection and the change of scale multiply the area of the inner triangle OXY and the total area of OLR by the same factor. Thus if $\hat{X}$ and $\hat{Y}$ denote the images of X and Y under the projection and scale change, we have $\text{Area } OXY/\text{Area } OLR = \text{Area } O'\hat{X}\hat{Y}/\text{Area } O'L'R'$. The projection and scale change preserve the uniform distribution of X and Y, that is, $\hat{X}$ and $\hat{Y}$ are randomly distributed in $O'L'R'$ and have the same distribution as X' and Y'. Hence

$$\frac{E(\text{Area } OXY)}{\text{Area } OLR} = \frac{E(\text{Area } O'X'Y')}{\text{Area } O'L'R'}.$$

We shall now use this fact to derive $E(\text{Area } OXY)$ for two random points X and Y falling in a triangle OLR. Let M denote the midpoint of the side LR and

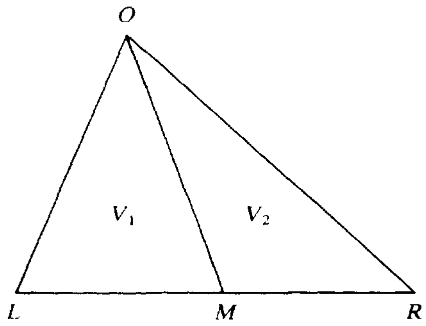


FIG. 5.4

let V_1 and V_2 denote the areas to the left and right, respectively, of the median OM. (See Fig. 5.4.) Then we can write

$$\begin{aligned} E(\text{Area } OXY) = {} & E(\text{Area } OXY | X, Y \in V_1)(\Pr[X \in V_1])^2 \\ & + E(\text{Area } OXY | X, Y \in V_2)(\Pr[X \in V_2])^2 \\ & + E(\text{Area } OXY | X \in V_1, Y \in V_2) \Pr[X \in V_1] \Pr[Y \in V_2] \\ & + E(\text{Area } OXY | X \in V_2, Y \in V_1) \Pr[X \in V_2] \Pr[Y \in V_1]. \end{aligned} \tag{5.41}$$

Since OM bisects the triangle,

$$\Pr[X \in V_1] = \Pr[X \in V_2] = \frac{1}{2} \tag{5.42}$$

and

$$\begin{aligned} E(\text{Area } OXY) = \frac{1}{4}[& E(\text{Area } OXY | X, Y \in V_1) \\ & + E(\text{Area } OXY | X, Y \in V_2) \\ & + 2E(\text{Area } OXY | X \in V_1, Y \in V_2)]. \end{aligned} \tag{5.43}$$

Since the areas of V_1 and V_2 are each one half the area of OLR, we have

$$\frac{E(\text{Area } OXY)}{\text{Area } OLR} = \frac{E(\text{Area } OXY | X, Y \in V_2)}{\frac{1}{2} \text{Area } OLR} \tag{5.44}$$

which implies

$$E(\text{Area } OXY | X, Y \in V_i) = \frac{1}{2} E(\text{Area } OXY) \tag{5.45}$$

for $i=1,2$. Substituting $\frac{1}{2}E(\text{Area } OXY)$ for $E(\text{Area } OXY \mid X, Y \in V_1)$ and $E(\text{Area } OXY \mid X, Y \in V_2)$ in our equation for $E(\text{Area } OXY)$ we get

$$
\begin{aligned}
E(\text{Area } OXY) = \frac{1}{4}\Big[\frac{1}{2}E(\text{Area } OXY) &+ \frac{1}{2}E(\text{Area } OXY) \\
&+ 2E(\text{Area } OXY \mid X \in V_1, Y \in V_2)\Big]
\end{aligned}
\tag{5.46}
$$

which yields

$$
E(\text{Area } OXY) = \frac{2}{3}E(\text{Area } OXY \mid X \in V_1, Y \in V_2) = \frac{2}{3}\text{Area } O\Gamma_1\Gamma_2 \tag{5.47}
$$

where Γ_1 and Γ_2 denote the centers of gravity of V_1 and V_2 respectively. We have already found the area of a triangle of this type to be

$$
\text{Area } O\Gamma_1\Gamma_2 = \frac{2}{9}\text{Area } OLR. \tag{5.48}
$$

Therefore

$$
E(\text{Area } OXY) = \frac{4}{27}\text{Area } OLR. \tag{5.49}
$$

Applying this result to the two triangles S_1 and S_2 in our original triangle ABC we get

$$
\begin{aligned}
E_O(\text{Area } OXY \mid X, Y \in S_1) &= \frac{4}{27}s_1, \\
E_O(\text{Area } OXY \mid X, Y \in S_2) &= \frac{4}{27}s_2.
\end{aligned}
\tag{5.50}
$$

We can now return to our expression for the expectation of the area of OXY conditioned on the position of O on the boundary of the triangle ABC.

$$
\begin{aligned}
E_O(\text{Area } OXY) &= E_O(\text{Area } OXY \mid X, Y \in S_1)s_1^2/A_D^2 \\
&\quad + E_O(\text{Area } OXY \mid X, Y \in S_2)s_2^2/A_D^2 \\
&\quad + 2E_O(\text{Area } OXY \mid X \in S_1, Y \in S_2)s_1s_2/A_D^2 \\
&= \frac{1}{A_D^2}\left[\frac{4}{27}s_1^3 + \frac{4}{27}s_2^3 + \frac{2}{9}A_Ds_1s_2\right].
\end{aligned}
\tag{5.51}
$$

Up to now we have considered O as a fixed point on the boundary of the triangle ABC. Now we will average over all possible positions for O on the boundary in order to obtain the unconditional expectation of the area of the triangle OXY, which we can then use to find Pr [RQ], the probability of four random points forming a re-entrant quadrilateral. We need average only over

possible positions for O on one side, say BC, of the triangle because the situation is identical for O on each of the other sides, AB and AC.

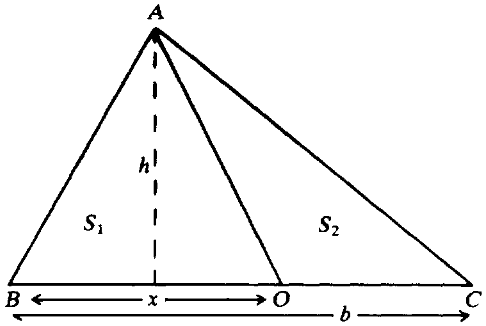


FIG. 5.5

Now suppose the length BC is b and the point O lies at a distance x from B. Furthermore, denote the height of A above the base BC by h. (See Fig. 5.5.) Then

$$A_D=\frac{1}{2}bh, \quad s_1=\frac{1}{2}xh, \quad s_2=\frac{1}{2}(b-x)h, \tag{5.52}$$

and the expected area of OXY regardless of the position of O on the boundary is

$$\begin{aligned} E(OXY\,|\,O\in\delta D) &= \frac{4}{b^2h^2}\int_0^b\left[\frac{1}{54}x^3h^3+\frac{1}{54}(b-x)^3h^3+\frac{1}{36}bx(b-x)h^3\right]\frac{dx}{b} \\ &= \frac{4}{b^3h^2}\left[\frac{b^4h^3}{216}+\frac{b^4h^3}{216}+\frac{b^4h^3}{72}-\frac{b^4h^3}{108}\right] \\ &= \frac{1}{9}\left(\frac{1}{2}bh\right)=\frac{1}{9}A_D. \end{aligned} \tag{5.53}$$

Thus the probability of a re-entrant quadrilateral being formed from four random points in a triangle is

$$\begin{aligned} \Pr[\mathrm{RQ}] &= \Pr[\mathrm{RQ}\,|\,O\in\delta D] \\ &= 3E(OXY\,|\,O\in\delta D)/A_D \\ &= \frac{1}{3}. \end{aligned} \tag{5.54}$$

The method by which we have obtained Pr [RQ] for four random points in a triangle can be extended to derive Pr [RQ] for four random points in any convex polygon. Suppose the polygon has n sides. Then we can take a point O uniformly distributed on one of the sides. From O we can draw lines to each of the vertices of the polygon thus dividing the polygon into $n-1$ triangles,

$S_1, S_2, \cdots, S_{n-1}$ (see Fig. 5.6). For any fixed position for the point O we can calculate

$$E_O(\text{Area } OXY) = \sum_{i=1}^{n-1} E_O(\text{Area } OXY \mid X, Y \in S_i) \Pr[X, Y \in S_i]$$

(5.55)

$$+ \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} E_O(\text{Area } OXY \mid X \in S_i, Y \in S_j) \cdot \Pr[X \in S_i] \Pr[Y \in S_j].$$

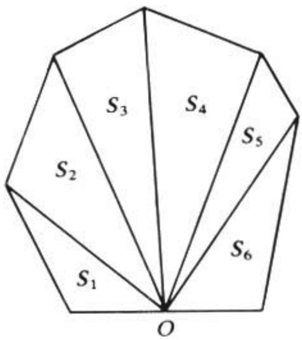


FIG. 5.6

Alikoski (1939) has solved the Sylvester problem for all regular n-gons employing the results obtained above for the triangle. He proves that

$$\Pr[\text{RQ}] = \frac{9\cos^2\omega + 52\cos\omega + 44}{9n^2\sin^2\omega} \quad \text{and} \quad \omega = \frac{2\pi}{n}. \tag{5.56}$$

We now present his development. Let $P_0, P_1, \cdots, P_{n-1}$ be a regular n-gon, denoted together with its area by T. Choose a rectangular coordinate system so that the polygon lies above the x-axis, the side $P_{n-1}P_0$ is on the x-axis, its midpoint is the origin. (See Fig. 5.7.) Let (x_ν, y_ν) be the coordinates of P_ν. We choose a point P of abscissa x on the side $P_{n-1}P_0$ and denote by $T_{\mu,\nu}(x)$ the triangle $PP_\mu P_\nu$ and its area. Now take two points X and Y in T at random and let $M(x)$ be the mean of the area of the triangle PXY. We further denote $M_\nu(x)$ the mean area of the triangle PXY when only those triangles are included for which both X and Y belong to $T_{\nu-1,\nu}(x)$, and $M_{\mu,\nu}(x)$ the mean area when one is in $T_{\mu-1,\mu}(x)$ and the other in $T_{\nu-1,\nu}(x)$.

We can now write

$$M(x) = \sum_{\nu=1}^{n-1} \left(\frac{T_{\nu-1,\nu}(x)}{T}\right)^2 \cdot M_\nu(x) + 2\sum_{\nu=2}^{n-1}\sum_{\mu=1}^{\nu-1} \frac{T_{\mu-1,\mu}(x)}{T} \cdot \frac{T_{\nu-1,\nu}(x)}{T} \cdot M_{\mu,\nu}(x). \tag{5.57}$$

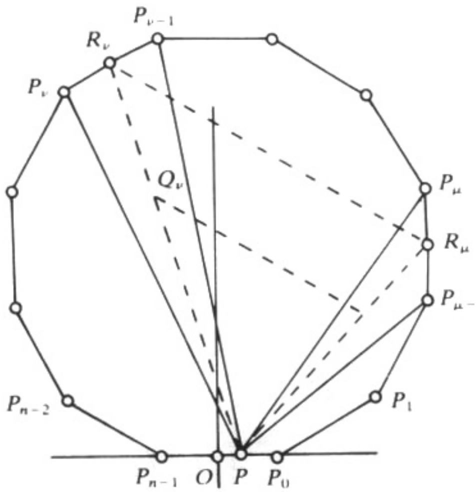


FIG. 5.7

If Q_ν is the center of gravity of the triangle $T_{\nu-1,\nu}(x)$ and R_ν the midpoint of $P_{\nu-1}P_\nu$, then we have from our previous results for the triangle,

$$M_\nu(x)=\frac{4}{27}\cdot T_{\nu-1,\nu}(x),$$

$$M_{\mu,\nu}(x)=\Delta PQ_\mu Q_\nu=\frac{4}{9}\cdot \Delta PR_\mu R_\nu.$$

Denoting

(5.58) $$\tilde{T}_{\mu,\nu}(x)=\Delta PR_\mu R_\nu,$$

we can write (5.57) as

(5.59) $$\begin{aligned}T^2\cdot M(x)=&\frac{4}{27}\sum_{\nu=1}^{n-1}[T_{\nu-1,\nu}(x)]^3\\&+\frac{8}{9}\sum_{\nu=2}^{n-1}\sum_{\mu=1}^{\nu-1}T_{\mu-1,\nu}(x)\cdot T_{\nu-1,\nu}(x)\cdot \tilde{T}_{\mu,\nu}(x).\end{aligned}$$

Using the expression of the area of a triangle as a determinant, we get ($\mu<\nu$)

$$T_{\mu,\nu}(x)=\frac{1}{2}\begin{vmatrix}1&1&1\\x&x_\mu&x_\nu\\0&y_\mu&y_\nu\end{vmatrix}\quad\text{or}\quad T_{\mu,\nu}(x)=\frac{1}{2}(A_{\mu,\nu}+x\cdot B_{\mu,\nu}),$$

where

(5.60) $$A_{\mu,\nu}=\begin{vmatrix}x_\mu&x_\nu\\y_\mu&y_\nu\end{vmatrix},\qquad B_{\mu,\nu}=y_\mu-y_\nu.$$

Similarly one gets:

$$\bar{T}_{\mu,\nu}(x)=\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & \frac{1}{2}(x_{\mu-1}+x_\mu) & \frac{1}{2}(x_{\nu-1}+x_\nu) \\ 0 & \frac{1}{2}(y_{\mu-1}+y_\mu) & \frac{1}{2}(y_{\nu-1}+y_\nu) \end{vmatrix}$$

or

$$\bar{T}_{\mu,\nu}(x)=\frac{1}{8}(\bar{A}_{\mu,\nu}+x\cdot\bar{B}_{\mu,\nu}),$$

where

$$\begin{aligned} \bar{A}_{\mu,\nu}&=A_{\mu-1,\nu-1}+A_{\mu,\nu-1}+A_{\mu-1,\nu}+A_{\nu,\nu}, \\ \bar{B}_{\mu,\nu}&=B_{\mu-1,\nu-1}+B_{\mu,\nu-1}+B_{\mu-1,\nu}+B_{\mu,\nu}. \end{aligned} \tag{5.61}$$

Noticing that $\bar{T}_{\nu,\nu}(x)$ is identically zero, we can write (5.59) in the following form:

$$\begin{aligned} T^2\cdot M(x)=&\frac{1}{54}\sum_{\nu=1}^{n-1}(A_{\nu-1,\nu}+x\cdot B_{\nu-1,\nu})^3 \\ &+\frac{1}{36}\sum_{\nu=1}^{n-1}\sum_{\mu=1}^{\nu}(A_{\nu-1,\nu}+x\cdot B_{\nu-1,\nu})(A_{\mu-1,\mu}+x\cdot B_{\mu-1,\mu}) \\ &\cdot(\bar{A}_{\mu,\nu}+x\cdot\bar{B}_{\mu,\nu}). \end{aligned} \tag{5.62}$$

The right hand side in (5.62) is a polynomial in x of the 3rd degree at most. But since $M(-x)=M(x)$ by symmetry, the terms of even degree must be zero, so that we may write

$$T^2\cdot M(x)=\alpha+\beta x^2. \tag{5.63}$$

A simple transformation yields:

$$\begin{aligned} \alpha&=\frac{1}{54}\sum_{\nu=1}^{n-1}A^3_{\nu-1,\nu}+\frac{1}{36}\sum_{\nu=1}^{n-1}A_{\nu-1,\nu}\sum_{\mu=1}^{\nu}A_{\mu-1,\mu}\bar{A}_{\mu,\nu}, \\ \beta&=\frac{1}{18}\sum_{\nu=1}^{n-1}A_{\nu-1,\nu}B^2_{\nu-1,\nu}+\frac{1}{36}\sum_{\nu=1}^{n-1}B_{\nu-1,\nu}\sum_{\mu=1}^{\nu}(B_{\mu-1,\mu}\bar{A}_{\mu,\nu}+2A_{\mu-1,\mu}\bar{B}_{\mu,\nu}). \end{aligned} \tag{5.64}$$

The expression for α comes directly from (5.62) while that for β can be deduced from (5.62) as follows. From (5.62) it follows that

$$\begin{aligned} \beta=&\frac{1}{18}\sum_{\nu=1}^{n-1}A_{\nu-1,\nu}B^2_{\nu-1,\nu}+\frac{1}{36}\sum_{\nu=1}^{n-1}\sum_{\mu=1}^{\nu}B_{\nu-1,\nu}B_{\mu-1,\mu}\bar{A}_{\mu,\nu} \\ &+\frac{1}{36}\sum_{\nu=1}^{n-1}\sum_{\mu=1}^{\nu}A_{\nu-1,\nu}B_{\mu-1,\mu}\bar{B}_{\mu,\nu}+\frac{1}{36}\sum_{\nu=1}^{n-1}\sum_{\mu=1}^{\nu}B_{\nu-1,\nu}A_{\mu-1,\mu}\bar{B}_{\mu,\nu}. \end{aligned} \tag{5.65}$$

In order to get the expression above for β we have to show that the last two sums—which we denote by S_1 and S_2—are equal to one another. In fact we have

$$S_2 = \sum_{\mu=1}^{n-1} \sum_{\nu=\mu}^{n-1} B_{\nu-1,\nu} A_{\mu-1,\nu} \bar{B}_{\nu,\mu}. \tag{5.66}$$

But since $\bar{B}_{\mu,\nu} = -\bar{B}_{\nu,\mu}$

$$\sum_{\mu=1}^{n-1} \sum_{\nu=\mu}^{\mu} B_{\nu-1,\nu} A_{\mu-1,\mu} \bar{B}_{\mu,\nu} = 0 \tag{5.67}$$

so that, since $\bar{B}_{\mu,\nu} = 0$,

$$S_2 = -\sum_{\mu=1}^{n-1} \sum_{\nu=1}^{\mu} B_{\nu-1,\nu} A_{\mu-1,\mu} \bar{B}_{\mu,\nu}. \tag{5.68}$$

Interchanging the letters μ and ν, and noting that $\bar{B}_{\nu,\mu} = -\bar{B}_{\mu,\nu}$, we get $S_2 = S_1$.

Let $\bar{M}$ be the mean of $M(x)$, when P is a randomly chosen point on $P_{n-1}P_0$. We get:

$$\bar{M} = \frac{1}{x_0} \int_0^{x_0} M(x)\, dx, \tag{5.69}$$

that is

$$\bar{M} = \frac{1}{3T^2}(3\alpha + \beta x_0^2). \tag{5.70}$$

If we let M be the mean of the area of the triangle XYZ where XYZ are randomly chosen inside the polygon, then it can be shown

$$4M = 3\bar{M} \quad \text{or} \quad 4M = \frac{3\alpha + \beta x_0^2}{T^2}.$$

Therefore we get:

$$\Pr[\mathrm{RQ}] = \frac{3\alpha + \beta x_0^2}{T^3}. \tag{5.71}$$

We chose the radius of the circle circumscribed to the polygon as unit length and so we get

$$\varphi = \frac{\pi}{n}. \tag{5.72}$$

Then the coordinates of P_ν are

$$x_\nu = \sin(2\nu+1)\varphi, \qquad y_\nu = \cos\varphi - \cos(2\nu+1)\varphi. \tag{5.73}$$

Upon putting those values in (5.60), we get:

$$\begin{aligned} A_{\mu,\nu} &= \cos\varphi[\sin(2\mu+1)\varphi - \sin(2\nu+1)\varphi] + \sin 2(\nu-\mu)\varphi, \\ B_{\mu,\nu} &= \cos(2\nu+1)\varphi - \cos(2\mu+1)\varphi, \end{aligned} \tag{5.74}$$

and hence

$$A_{\mu-1,\mu} = 2 \sin 2\varphi \sin^2 \mu\varphi, \quad B_{\mu-1,\mu} = -2 \sin \varphi \sin 2\mu\varphi. \tag{5.75}$$

Upon substituting these values in (5.61), after simple transformations we get:

$$\bar{A}_{\mu,\nu} = 16 \cos^2 \varphi \sin \nu\varphi \sin \mu\varphi \sin (\nu-\mu)\varphi, \quad \bar{B}_{\mu,\nu} = 4 \cos \varphi(\cos 2\nu\varphi - \cos 2\mu\varphi). \tag{5.76}$$

In order to compute φ we put these values in the above expressions for α and β and then write (5.71) in the form

$$\Pr[\mathrm{RQ}] = \frac{8(3\alpha + \beta \sin^2 \varphi)}{n^3 \cdot \sin^3 2\varphi}.$$

Using De Moivre's formula, we see the result for Pr [RQ] becomes (5.56).

In order to show that Pr [RQ] decreases with increasing n, take the derivative of Pr [RQ] with respect to ω and we get

$$\frac{dp}{d\omega} = \frac{\omega}{18\pi^2 \sin^3 \omega} \cdot S, \tag{5.77}$$

where

$$S = \sin \omega(9 \cos^2 \omega + 52 \cos \omega + 44) - \omega(26 \cos^2 \omega + 53 \cos \omega + 26). \tag{5.78}$$

A power series expansion yields:

$$S = \omega^5 - \frac{5}{12}\omega^7 + \frac{3}{40}\omega^9 - + \cdots. \tag{5.79}$$

This is an alternating series and since for $n \geqq 5$ ($\omega \leqq 2\pi/5$) the absolute value of its terms is decreasing, we get for these values of n:

$$S > \omega^5 - \frac{5}{12}\omega^7. \tag{5.80}$$

Since the right hand side, and consequently $d \Pr[\mathrm{RQ}]/d\omega$ is positive for $n \geqq 5$, Pr [RQ] decreases as n increases beyond $n = 5$. Noting that the values of Pr [RQ] are also decreasing for $n = 3, 4, 5$, we find the assertion is proved.

If we let $n \to \infty$, then $\omega \to 0$ and $\Pr[\mathrm{RQ}] \to 35/(12\pi^2)$. The following values (Table 3) for Pr [RQ] had been determined by other investigators previously but substitution for $n = 3, 4, 6, 8, \infty$ in the Alikoski result will provide the same answers.

Some other problems in geometrical probability can be solved by noticing that their solutions depend on the solution to the Sylvester four point problem. For example, if two points A and B are taken at random in a convex area, suppose we seek the probability that two other random points C and D lie on opposite sides of the line AB. Once three points A, B and C have been placed, the

TABLE 3

	Triangle	Square (Parallelogram)	Regular Hexagon	Regular Octagon	Circle (Ellipse)
Pr[RQ]	$\frac{1}{3}$	$\frac{11}{36}$	$\frac{289}{972}$	$\frac{1181+867\sqrt{2}}{4032+2880\sqrt{2}}$	$\frac{35}{12\pi^2}$
or	.3333	.3056	.2973	.2970	.2955

situation is shown in Fig. 5.8. The quadrilateral formed by the four random points A, B, C and D is convex if and only if the fourth point D falls in an area marked CQ. Clearly the regions marked CQ have the same mean area by symmetry. There are two areas marked CQ on the same side of AB as C and one area on the other side. Thus if we know that the quadrilateral formed by A, B, C and D is convex, the probability is $\frac{1}{3}$ that the fourth random point falls on the opposite side of AB from C.

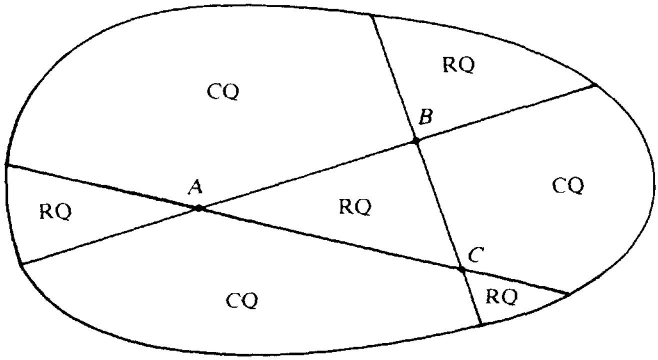


FIG. 5.8

On the other hand, the quadrilateral $ABCD$ is re-entrant if and only if the fourth point D falls in an area marked RQ. It does not seem intuitively obvious that the regions marked RQ have the same mean area, but this fact can be shown by the following considerations. If the fourth point falls in the topmost region marked RQ then the point B will be the re-entrant vertex of the quadrilateral. If the fourth point falls in the central region marked RQ, it will itself be the re-entrant vertex of the quadrilateral. Each one of the random points is equally likely to be the re-entrant vertex of the quadrilateral. Thus the mean areas of the regions marked RQ must be the same. There are two areas marked RQ on each side of the line AB. Thus the probability is $\frac{1}{2}$ that the fourth point D falls on the

opposite side of the line AB from the third point C. Hence the probability that C and D fall on opposite sides of the line AB is

$$\begin{aligned}\Pr[\text{opposite sides of } AB] &= \Pr[\text{opposite sides}|\text{convex}]\Pr[\text{convex}] \\ &\quad +\Pr[\text{opposite sides}|\text{re-entrant}]\Pr[\text{re-entrant}] \\ &= \frac{1}{3}(1-\Pr[\text{RQ}])+\frac{1}{2}\Pr[\text{RQ}] \\ &= \frac{1}{3}+\frac{1}{6}\Pr[\text{RQ}]\end{aligned} \tag{5.81}$$

where Pr [RQ], the probability of a re-entrant quadrilateral being formed from four random points in a convex region, is the solution of the Sylvester four point problem.

We have just shown that if K is any convex region and Q_1, Q_2, Q_3, and Q_4 are four random points in K, the probability Pr [RQ] that the four points form a re-entrant quadrilateral satisfies the relation

$$\Pr[\text{RQ}] = 4-6\Pr[\text{same side of } Q_1Q_2] \tag{5.82}$$

where Pr [same side of Q_1Q_2] denotes the probability that Q_3 and Q_4 fall on the same side of the line Q_1Q_2.

To evaluate Pr [same side of Q_1Q_2], suppose the line Q_1Q_2 has direction θ and distance p from some origin. This line splits the area A of K into two sections with areas, say, $\sigma(p, \theta)$ and $\sigma'(p, \theta)$. If $f(p, \theta)$ is the joint density of the random variables p and θ, then

$$\Pr[\text{same side of } Q_1Q_2] = \int\int\left[\frac{\sigma^2(p,\theta)}{A^2}+\frac{\sigma'^2(p,\theta)}{A^2}\right]f(p,\theta)\,dp\,d\theta \tag{5.83}$$

which, by the symmetry between the roles of $\sigma(p, \theta)$ and $\sigma'(p, \theta)$, reduces to

$$\Pr[\text{same side of } Q_1Q_2] = \frac{2}{A^2} = \int\int \sigma^2(p,\theta)f(p,\theta)\,dp\,d\theta. \tag{5.84}$$

We shall obtain $f(p, \theta)$ as the limit as $\Delta p \to 0$ and $\Delta\theta \to 0$ of $(\Delta\theta\,\Delta p)^{-1}$ times the probability that the line Q_1Q_2 has direction between θ and $\theta+\Delta\theta$ and distance between p and $p+\Delta p$ from the origin. For any given position of Q_1, the line joining Q_1 and Q_2 has a direction between θ and $\theta+\Delta\theta$ if and only if Q_2 falls in the shaded region shown in Fig. 5.9. This happens with probability

$$\Pr[\text{direction } Q_1Q_2 \in (\theta, \theta+\Delta\theta)|Q_1] = \frac{\Delta\theta}{A}\left[\frac{r^2}{2}+\frac{r'^2}{2}\right]. \tag{5.85}$$

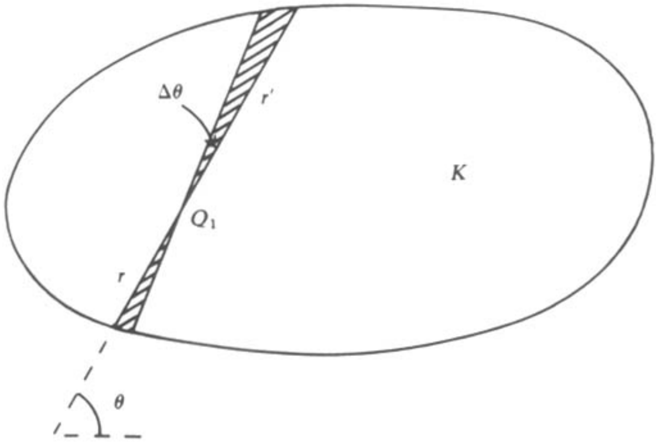


FIG. 5.9

Integrating this expression over a uniform probability distribution for all positions of Q_1 in the shaded strip in Fig. 5.10, and multiplying by the probability that Q_1 falls in the strip $(c(p, \theta)\,\Delta p/A)$, we get

$$\Pr\begin{bmatrix} Q_1Q_2 \text{ has direction between} \\ \theta \text{ and } \theta+\Delta\theta \text{ and distance} \\ \text{between } p \text{ and } p+\Delta p \text{ from origin}\end{bmatrix}$$

$$= \frac{\Delta\theta}{2A}\int_0^{c(p,\theta)} [r^2+(c(p,\theta)-r)^2]\frac{dr}{c(p,\theta)}\left(\frac{c(p,\theta)\,\Delta p}{A}\right) \tag{5.86}$$

$$= \frac{\Delta\theta\,\Delta p c^3(p,\theta)}{3A^2}.$$

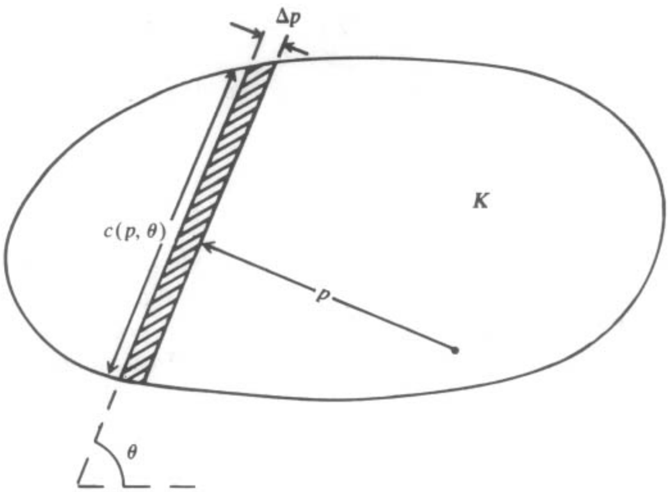


FIG. 5.10

Thus

$$f(p, \theta) = \frac{c^3}{3A^2} \tag{5.87}$$

and

$$\Pr[\text{same side of } Q_1Q_2] = \frac{2}{3A^4} \int\int \sigma^2(p, \theta)c^3(p, \theta)\, dp\, d\theta. \tag{5.88}$$

Calculating Pr [RQ] from Pr [same side of Q_1Q_2], we obtain the following:

THEOREM. *Three points are taken at random in a convex region K with area A. The mean area of the triangle formed by these three points is*

$$E(\Delta) = A - \frac{1}{A^3} \int\int \sigma^2(p, \theta)c^3(p, \theta)\, dp\, d\theta \tag{5.89}$$

where θ and p are the parameters of a line intersecting K, $c(p, \theta)$ is the chord length of that line within K, and $\sigma(p, \theta)$ is the area in K on one side of the line. If a fourth point is taken at random in K, the probability that the four random points form a re-entrant quadrilateral is

$$\Pr[\text{RQ}] = 4 - \frac{4}{A^4} \int\int \sigma^2(p, \theta)c^3(p, \theta)\, dp\, d\theta. \tag{5.90}$$

Sylvester's problem in three dimensions. Results analogous to those presented in the preceding section were obtained by Bohnslav Hostinsky (1925) for the three dimensional analogue to Sylvester's problem. Consider five points Q_1, Q_2, Q_3, Q_4 and Q_5 independently and uniformly distributed over the volume of a convex region K in three-dimensional space. What is the probability that one of the points falls within the tetrahedron formed by the other four points? To obtain an expression for this probability, we shall trace Hostinsky's steps and shall first derive the probability that Q_4 and Q_5 fall on the same side of the plane passing through Q_1, Q_2 and Q_3.

To calculate this integral, let us change our variables of integration from

$$X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$$

to the new variables

$$x_1', x_2', x_3', y_1', y_2', y_3', u, v, w$$

where

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3, \quad y_1' = y_1, \quad y_2' = y_2, \quad y_3' = y_3$$

and u, v and w are the parameters of the plane

$$ux + vy + wz + 1 = 0$$

passing through the points Q_1, Q_2 and Q_3. In other words, u, v and w are the

solution of the linear system of equations

$$
\begin{aligned}
&ux_1+vy_1+wz_1+1=0,\\
(5.91)\qquad &ux_2+vy_2+wz_2+1=0,\\
&ux_3+vy_3+wz_3+1=0.
\end{aligned}
$$

The Jacobian of this transformation reduces to

$$
(5.92)\qquad \begin{vmatrix}
\dfrac{-x_1'}{w} & \dfrac{-x_2'}{w} & \dfrac{-x_3'}{w}\\[2ex]
\dfrac{-y_1'}{w} & \dfrac{-y_2'}{w} & \dfrac{-y_3'}{w}\\[2ex]
\dfrac{1}{w^2}[ux_1'+vy_1'+1] & \dfrac{1}{w^2}[ux_2'+vy_2'+1] & \dfrac{1}{w^2}[ux_3'+vy_3'+1]
\end{vmatrix}
$$

which, in turn, reduces to

$$
(5.93)\qquad \frac{1}{w^4}\begin{vmatrix} x_1' & x_2' & x_3'\\ y_1' & y_2' & y_3'\\ 1 & 1 & 1\end{vmatrix}
$$

or

$$
\frac{1}{w^4}|x_1'y_2'-x_2'y_1'+x_2'y_3'-x_3'y_2'+x_3'y_1'-x_1'y_3'|.
$$

It is a well-known fact in analytical geometry that the area of a triangle whose vertices are (x_1', y_1'), (x_2', y_2') and (x_3', y_3') is

$$
(5.94)\qquad \Delta p=\tfrac{1}{2}|x_1'y_2'-x_2'y_1'+x_2'y_3'-x_3'y_2'+x_3'y_1'-x_1'y_3'|.
$$

Thus our element of integration $dx_1\,dy_1\,dz_1\cdots dx_3\,dy_3\,dz_3$ becomes

$$
(5.95)\qquad \frac{2\,\Delta p}{w^4}\,dx_1'\,dy_1'\cdots dy_3'\,du\,dv\,dw
$$

where Δp is the area projected onto the X-Y plane by the triangle $Q_1Q_2Q_3$. Multiplying and dividing by $(u^2+v^2+w^2)^2$, we have

$$
(5.96)\qquad 2\left(\frac{\sqrt{u^2+v^2+w^2}}{w}\right)^4\Delta p\frac{dx_1'\,dy_1'\cdots dy_3'\,du\,dv\,dw}{(u^2+v^2+w^2)^2}.
$$

We note that $du\,dv\,dw/(u^2+v^2+w^2)^2$ is the density for planes in space which is invariant under rigid motions of the coordinate axes. We can rewrite this density as $\sin\varphi\,dp\,d\varphi\,d\theta$ where (p,φ,θ) are the spherical coordinates of the closest point on the plane to the origin.

Then

$$
(5.97)\qquad u=\frac{\sin\varphi\cos\theta}{p},\quad v=\frac{\sin\varphi\sin\theta}{p},\quad w=\frac{\cos\varphi}{p}
$$

and

$$\left(\frac{\sqrt{u^2+v^2+w^2}}{w}\right)^4=\frac{1}{\cos^4\varphi}. \tag{5.98}$$

Since φ is the angle between the normal to the plane and the z-axis, we see from Fig. 5.11 that any increment $d_\xi d_y$ of area on the plane projects onto the x-y plane as $dx\,dy/\cos\varphi$. Thus our element of integration

$$\frac{2\Delta p}{\cos\varphi}\,\frac{dx_1'\,dy_1'}{\cos\varphi}\,\frac{dx_2'\,dy_2'}{\cos\varphi}\,\frac{dx_3'\,dy_3'}{\cos\varphi}\sin\varphi\,dp\,d\varphi\,d\theta \tag{5.99}$$

becomes

$$2\Delta\,d\xi_1\,dy_1\,d\xi_2\,dy_2\,d\xi_3\,dy_3\sin\varphi\,dp\,d\varphi\,d\theta \tag{5.100}$$

where Δ is the area of the triangle $Q_1Q_2Q_3$; p, φ, and θ are the coordinates of the plane determined by Q_1, Q_2, and Q_3; and ξ_k and y_k are the coordinates of the point Q_k ($k=1,2,3$) with respect to a system of coordinates situated in the plane $Q_1Q_2Q_3$.

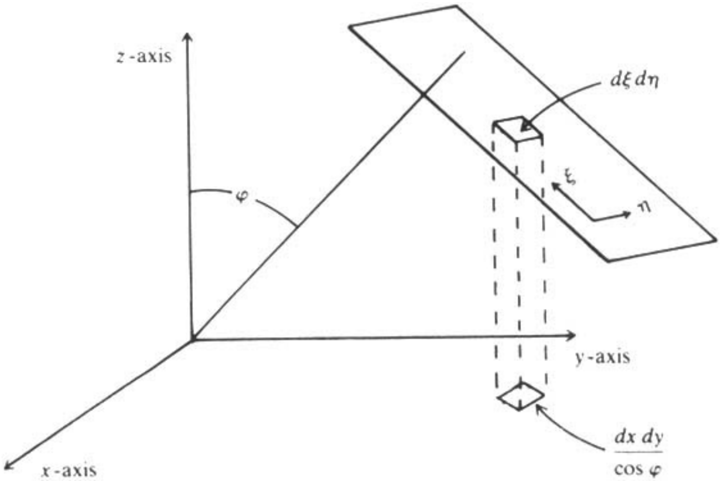


FIG. 5.11

The plane $Q_1Q_2Q_3$ cuts K into two parts with volumes, say, Σ and Σ'. The measure of all positions of Q_4 and Q_5 such that they fall on the same side of the plane $Q_1Q_2Q_3$ is

$$\iiint\iiint dx_4\,dy_4\,dz_4\,dx_5\,dy_5\,dz_5=\Sigma^2+\Sigma'^2 \tag{5.101}$$

and, since $dx_1\,dy_1\cdots dz_3$ is

$$2\Delta\,d\xi_1\,dy_1\,d\xi_2\,dy_2\,d\xi_3\,dy_3\sin\varphi\,dp\,d\varphi\,d\theta, \tag{5.102}$$

the probability that Q_4 and Q_5 fall on the same side of the plane $Q_1Q_2Q_3$ can be written (where P denotes the probability)

$$P[\text{same side of } Q_1Q_2Q_3]$$

$$(5.103)\qquad \begin{aligned} &= \frac{1}{V^5}\iint\cdots\int dx_1\,dy_1\cdots dz_5 \\ &= \frac{2}{V^5}\iiint\iiint\iiint \Delta(\Sigma^2+\Sigma'^2)\,d\xi_1\,dy_1\,d\xi_2\,dy_2\,d\xi_3\,dy_3\,dp\,dw \end{aligned}$$

where dw denotes the increment in solid angle $\sin\varphi\,d\varphi\,d\theta$.

The value of $\Sigma^2+\Sigma'^2$ depends only on the position of the plane $Q_1Q_2Q_3$. In other words, the term $(\Sigma^2+\Sigma'^2)$ can be factored out of the integrations with respect to $d\xi_i$ and dy_i. The integral

$$(5.104)\qquad \iiint\iiint \Delta\,d\xi_1\,dy_1\,d\xi_2\,dy_2\,d\xi_3\,dy_3$$

is just $A^3E(\Delta)$, where $E(\Delta)$ is the mean area of a triangle formed by three random points in the area A of the convex section of the plane $Q_1Q_2Q_3$ within K. We have already obtained a formula for $E(\Delta)$, namely

$$(5.105)\qquad E(\Delta)=A-\frac{1}{A^3}\iint c^3\sigma^2\,dq\,d\psi$$

where σ is the area of one of the parts into which the area A is divided by the line (q, ψ), and c is the length of that line within the area A. The integration is extended over all secants of the area A. Hence we have

$$(5.106)\qquad \begin{aligned} P[\text{same side of } Q_1Q_2Q_3] &= \frac{2}{V^5}\iiint A^3E(\Delta)(\Sigma^2+\Sigma'^2)\,dp\,dw \\ &= \frac{2}{V^5}\iiint\left[A^4\iint \sigma^2c^3\,dq\,d\psi\right] \\ &\qquad\cdot[\Sigma^2+\Sigma'^2]\,dp\,dw \end{aligned}$$

where the integrations with respect to q and ψ extend over all lines in the plane (p, φ, θ) which hit the curve of intersection of that plane with K, and the integrations with respect to the coordinates of this plane extend over all planes cutting K.

We shall now proceed to find the mean volume of the tetrahedron T formed by four points Q_1, Q_2, Q_3 and Q_4 taken at random within K.

The planes containing the faces of the tetrahedron T divide the volume V of K into 15 parts. A first part T_0 is formed by the interior of the tetrahedron T. In order to designate the other parts, let τ_i represent the plane containing the face of T opposite the vertex Q_i, and define the negative side of τ_i to be the side on which the vertex Q_i is situated. For any permutation (i, j, k, m) of $(1, 2, 3, 4)$, we denote by T_i the part of K on the positive side of τ_i and on the negative sides of τ_j, τ_k, and τ_m; by T_{ij} the part of K on the positive sides of τ_i and τ_j and negative

sides of τ_k and τ_m; and by T_{ijk} the part of K on the positive sides of τ_i, τ_j and τ_k and the negative sides of τ_m. Using this notation, we can list the 15 parts of K

(5.107)

$$T_0;\quad T_1, T_2, T_3, T_4;\quad T_{12}, T_{23}, T_{31}, T_{14}, T_{24}, T_{34};\quad T_{123}, T_{124}, T_{134}, T_{234}.$$

If a fifth point is taken at random in K, designate by p_0, p_i, p_{ij} or p_{jk} the probabilities that the fifth point falls in T_0, T_i, T_{ij} or T_{ijk}, respectively. By symmetry, we have

$$\begin{aligned} &p_1 = p_2 = p_3 = p_4,\\ \text{(5.108)}\qquad &p_{12} = p_{23} = p_{31} = p_{14} = p_{24} = p_{34},\\ &p_{123} = p_{124} = p_{134} = p_{234}, \end{aligned}$$

and, since the fifth point must fall in one of the 15 sections of K,

$$\text{(5.109)}\qquad p_0 + 4p_1 + 6p_{12} + 4p_{123} = 1.$$

Let Q_4 be a fifth random point in K, and let U be the tetrahedron with vertices Q_1, Q_2, Q_3 and Q_5. The volume of K is divided by the planes of the faces of the tetrahedron U into 15 parts which we can represent

(5.110)

$$U_0, U_1, U_2, U_3, U_5;\ U_{12}, U_{23}, U_{31}, U_{15}, U_{25}, U_{35};\ U_{123}, U_{125}, U_{135}, U_{235},$$

where U_i, U_{ij}, and U_{ijk} are defined analogously to T_i, T_{ij} and T_{ijk}.

We can obtain relations between the U's and T's in the following way. From Fig. 5.12 we see that if Q_4 and Q_5 are on the same side of the plane $Q_1Q_2Q_3$, then Q_5 and Q_1 fall on opposite sides of the plane $Q_2Q_3Q_4$ if and only if Q_4 and Q_1 fall on the same side of the plane $Q_2Q_3Q_5$. By symmetry, the same statement is true with Q_1 interchanged with Q_2 or Q_3. Hence Q_5 is in T_{123} (i.e., on the opposite side of $Q_2Q_3Q_4$ from Q_1, the opposite side of $Q_1Q_3Q_4$ from Q_2, the opposite side of $Q_1Q_2Q_4$ from Q_3 and the same side of $Q_1Q_2Q_3$ as Q_4) if and only if Q_4 is in U_0 (i.e., on the same side of $Q_2Q_3Q_5$ as Q_1, the same side of

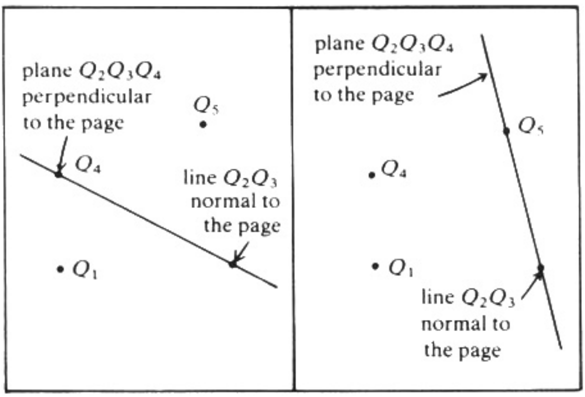


FIG. 5.12

$Q_1Q_3Q_5$ as Q_2, the same side of $Q_1Q_2Q_5$ as Q_3 and the same side of $Q_1Q_2Q_3$ as Q_4). Thus

$$p_{123} = p_0. \tag{5.111}$$

Similarly Q_5 is in T_1 (on the opposite side of $Q_2Q_3Q_4$ from Q_1, the same side of $Q_1Q_3Q_4$ as Q_2, the same side of $Q_1Q_2Q_4$ as Q_3, and the same side of $Q_1Q_2Q_3$ as Q_4). Thus

$$p_1 = p_{23}. \tag{5.112}$$

Since the indices are interchangeable,

$$p_{12} = p_3$$

and, since by symmetry $p_3 = p_1$, we have

$$p_{12} = p_1.$$

Combining the three equations

$$p_0 + 4p_1 + 6p_{12} + 4p_{123} = 1, \quad p_{123} = p_0, \quad p_{12} = p_1, \tag{5.113}$$

we get

$$5p_0 + 10p_1 = 1. \tag{5.114}$$

The points Q_4 and Q_5 fall on the same side of the plane $Q_1Q_2Q_3$ if and only if Q_5 falls on the negative side of the plane τ_4, i.e., if and only if Q_5 falls in a region T_i, T_{ij} or T_{ijk} where neither i nor j nor k equals four. Thus

$$\begin{aligned} P[\text{same side of } Q_1Q_2Q_3] &= p_0 + p_1 + p_2 + p_3 + p_{12} + p_{23} + p_{31} + p_{123} \\ &= 2p_0 + 6p_1. \end{aligned} \tag{5.115}$$

Eliminating p_1 from the last two equations, we get

$$p_0 = \frac{3}{5} - P[\text{same side of } Q_1Q_2Q_3] \tag{5.116}$$

which, together with our expression for $P[\text{same side of } Q_1Q_2Q_3]$, leads to:

THEOREM. *The mean volume of the tetrahedron whose vertices are four random points in a convex region K of volume V is*

$$\begin{aligned} E(V_T) &= V\left[\frac{3}{5} - \frac{2}{V^5}\iiint A^3E(\Delta)(\Sigma^2 + \Sigma'^2)\,dp\,dw\right] \\ &= V\left[\frac{3}{5} - \frac{2}{V^5}\iiint \left[A^4 - \iint c^3\sigma^2\,dq\,d\psi\right][\Sigma^2 + \Sigma'^2]\right]dp\,dw \end{aligned} \tag{5.117}$$

where A is the area of the intersection of K with the plane having coordinates (p, w) [i.e., (p, φ, θ)], $E(\Delta)$ is the mean area of a triangle formed by three random points in that area, Σ and Σ' are the volumes of the parts into which the plane divides K, and the integrations over $dp\,dw$ extend over all planes intersecting K. In

the second expression c is the length of intersection of the area A with a line having coordinates (q, ψ) in the plane (p, w), σ is the area within A on one side of that line, and the integrations over $dq\, d\psi$ cover all lines in the plane (p) which intersect K. If five random points are taken within K, the probability that any one of the five points falls within the tetrahedron formed by the other four is

$$
\begin{aligned}
P[\mathrm{RP}] &= 3-\frac{10}{V^5}\iiint A^3E(\Delta)(\Sigma^2+\Sigma'^2)\,dp\,dw \\
&= 3-\frac{10}{V^5}\iiint\left[A^4-\iint c^3\sigma^2\,dq\,d\psi\right](\Sigma^2+\Sigma'^2)\,dp\,dw
\end{aligned}
\tag{5.118}
$$

(*where* RP *denotes re-entrant polyhedron*).

We can apply these general results to the case where the region K is a sphere of radius r. A plane at distance p from the center of the sphere intersects the sphere in a circle of radius $\sqrt{r^2-p^2}$ and area

$$A=\pi(r^2-p^2).$$

From our discussion of Sylvester's problem for random points in a circle, we recall that the mean area of a triangle formed by three random points in a circle is $35/(48\pi)$ times the square of the radius of the circle. Thus for three random points on the circle of intersection of the sphere with the plane at distance p from the sphere's center, the mean area of the triangle formed is

$$E(\Delta)=\frac{35}{48\pi}(r^2-p^2). \tag{5.119}$$

The volume Σ in the sphere on the side of the plane, say, away from the sphere's center is

$$\Sigma=\frac{1}{3}\pi(r-p)^2(2r+p). \tag{5.120}$$

Thus

$$
\begin{aligned}
\Sigma^2+\Sigma'^2 &= \frac{2}{9}\pi^2[8r^6-4r^3(r-p)^2(2r+p)+(r-p)^4(2r+p)^2] \\
&= \frac{2}{9}\pi^2[4r^6+9r^4p^2-6r^2p^4+p^6]
\end{aligned}
\tag{5.121}
$$

and the formula

$$P[\text{same side of } Q_1Q_2Q_3]=\frac{2}{V^5}\iiint E(\Delta)A^3(\Sigma^2+\Sigma'^2)\,dp\,dw \tag{5.122}$$

becomes

$$P[\text{same side of } Q_1Q_2Q_3] = \frac{4\pi}{(\frac{4}{3}\pi r^3)^5} \int_{-r}^{r} \left(\frac{35}{48\pi}\right)(r^2-p^2)[\pi(r^2-p^2)]^3$$

$$\cdot \left[\frac{2}{9}\pi^2(4r^6+9r^4p^2-6r^2p^4+p^6)\right] dp \tag{5.123}$$

$$= \frac{84}{143}.$$

Thus when K is a sphere, the mean volume of the tetrahedron formed by four random points in K is

$$E(V_T) = \frac{4}{3}\pi r^3\left(\frac{3}{5} - \frac{84}{143}\right) = \frac{4}{3}\pi r^3\left(\frac{9}{715}\right) = \frac{12\pi r^3}{715} \tag{5.124}$$

and the probability that five random points in a sphere form a re-entrant polyhedron is

$$P[\text{RP}] = 5E(V_T)\Big/\left(\frac{4}{3}\pi r^3\right) = \frac{9}{143}. \tag{5.125}$$

For convex regions other than the sphere, Hostinsky's formula for $P[\text{RP}]$ does not seem to be tractable. Explicit values of $P[\text{RP}]$ for random points in non-spherical regions such as tetrahedrons, parallelepipeds, etc., have apparently not yet been successfully calculated. On the other hand, from the relation

$$1 > P[\text{RP}] = 5p_0$$

we have

$$p_0 < \frac{1}{5}.$$

In other words, the mean volume of the tetrahedron whose vertices are four random points in K is less than one fifth the total volume of K, no matter what shape K has (so long as it is convex).

It should also be noted that the above bound on p_0 and the relation

$$p_0 = \frac{3}{5} - P[\text{same side of } Q_1Q_2Q_3] \tag{5.126}$$

yield the bounds

$$\frac{2}{5} < P[\text{same side of } Q_1Q_2Q_3] < \frac{3}{5} \tag{5.127}$$

regardless of the shape of K.

For the case of four random points being taken in a convex region K on the place, Sylvester conjectured and Alikoski proved that the probability $P[\text{RQ}]$

that the four points form a re-entrant quadrilateral is minimized when K is a circle and maximized when K is a triangle. The analogous conjecture for the three-dimensional problem is that the probability $P[\mathrm{RP}]$ that five random points in K form a re-entrant polyhedron is minimized when K is a sphere and maximized when K is a tetrahedron. This conjecture remains unproven.

Kingman (1969) considers a generalization of Sylvester's problem in n dimensions. If $n+2$ points are chosen independently at random inside K, then the convex hull of these points is a convex polytope with either $n+1$ or $n+2$ vertices. We want to know the probability of its having $n+1$ vertices. This probability is $(n+2)E\{\Delta(x_0, x_1, \cdots, x_n)\}/V$, where $\Delta(x_0, x_1, \cdots, x_n)$ is the volume of the convex polytope formed by the points $x_0, \cdots, x_n$. In particular, it is found that for an n-dimensional ball,

$$\frac{E\{\Delta(x_0, \cdots, x_n)\}}{V} = \binom{n+1}{\frac{1}{2}(n+1)} \Big/ \binom{(n+1)^2}{\frac{1}{2}(n+1)^2} 2^n. \tag{5.128}$$

If we substitute $n=3$, it follows that the probability that the convex hull of five points chosen uniformly inside a sphere has four vertices, is $9/143$ and confirms the Hostinsky result in three dimensions for the sphere.

This page intentionally left blank

CHAPTER 6

Random Chords in the Circle and the Sphere

Techniques have recently been developed whereby cells in mitosis may be photographed. For the normal human cell the resulting picture, called a karyograph, displays 22 pairs of like chromosomes plus two sex chromosomes, like in the female and unlike in the male. See Fig. 6.1. Abnormal cells may have more or less chromosomes than these. The positions of the chromosomes on a karyograph present an interesting problem in geometrical probability; namely, to what extent is the position of each member of a pair of chromosomes random in the cell nucleus with respect to the other member of the pair. For example, are the chromosomes of a pair located closer to each other in general than would be two points randomly dropped in the cell nucleus? This latter question was investigated by Barton, David and Fix (1963), who used data from 70 karyographs to test the hypothesis that the chromosomes of a pair are randomly distributed within the picture of the cell nucleus versus the alternative hypothesis that there is a force of mutual attraction (or repulsion) between the two chromosomes in each pair.

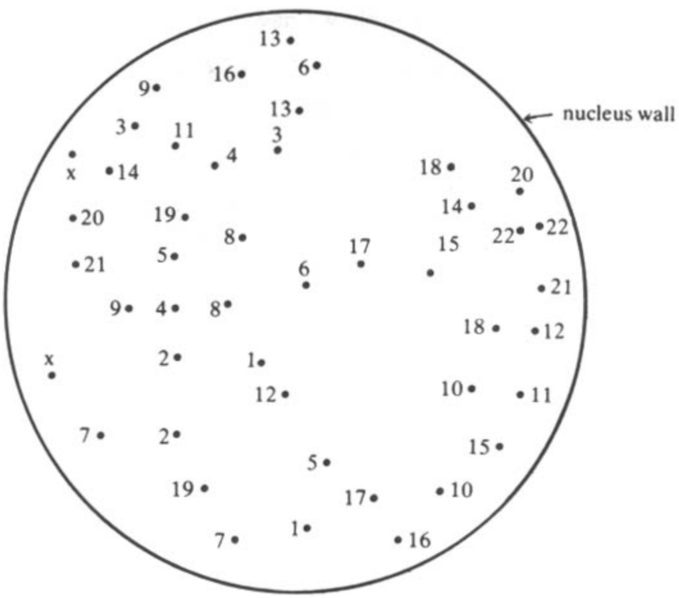


FIG. 6.1

To retrace the analysis done by Barton, David and Fix, we shall need to derive the distribution of the distance between two points randomly dropped on a picture of a cell nucleus. For simplicity we will assume the boundary of the cell nucleus in a karyograph is a circle. Actually the nucleus is only approximately spherical and its boundary often appears to the camera more like an ellipse. We shall make some allowances for this ellipticity later on in our analysis, but for the momemt we shall assume the boundary is a circle and shall derive the probability density function of the distance, x, between two points randomly dropped on a circle of radius R. Let P denote the probability that the two points are separated by a distance between x and $x+\Delta x$. Let P_1 denote the same probability when one of the points is known to be on the circumference of the circle. Then by Crofton's fixed point theorem we have

$$dP = 2(P_1 - P)V^{-1}\,dV \tag{6.1}$$

where V in this case is the area of the circle, that is, $V=\pi R^2$ and $dV = 2\pi R\,dR$. We see from Fig. 6.2 that when one point is on the circumference, the second point must fall in a section of an annulus (shaded in Fig. 6.2) for the two points to be separated by a distance between x and $x+\Delta x$. For infinitesimal Δx, the area of this annulus section is $2\phi x\,\Delta x$, where ϕ can be seen from the figure to be $\cos^{-1}(x/(2R))$. Hence the probability that the points are separated by a distance between x and $x+\Delta x$ is

$$P_1 = \frac{2x\,\Delta x\cos^{-1}(x/(2R))}{\pi R^2} \tag{6.2}$$

Substituting into Crofton's formula we have

$$dP = 2\left[\frac{2x\,\Delta x\cos^{-1}(x/(2R))}{\pi R^2} - P\right]\frac{2\,dR}{R}.$$

Rearranging terms we get

$$R\,dP + 4P\,dR = \frac{8x\,\Delta x\cos^{-1}(x/(2R))}{\pi R^2},$$

and multiplying both sides by R^3, we get

$$R^4\,dP + 4R^3P\,dR = \frac{8x\,\Delta xR\cos^{-1}(x/(2R))}{\pi},$$

and integrating both sides, we obtain

$$\begin{aligned} PR^4 &= \frac{4x^2\,\Delta x}{\pi}\int\frac{2R}{x}\cos^{-1}\left(\frac{x}{2R}\right)dR \\ &= \frac{x\,\Delta x}{\pi}\left[4R^2\cos^{-1}\left(\frac{x}{2R}\right) - 2xR\sqrt{1-\frac{x^2}{4R^2}}\right]+C, \end{aligned} \tag{6.3}$$

where C is a constant to be determined. For $R=x/2$, the two random points

would have to fall on the circumference diametrically across the circle from each other. This is, of course, a zero probability event. Thus when $R = x/2$, $p = 0$. This implies $C = 0$. Thus we have derived the probability density for the distance between two random points dropped in a circle:

$$p(x) = \frac{2x}{R^2}\left[\frac{2}{\pi}\cos^{-1}\left(\frac{x}{2R}\right) - \frac{x}{\pi R}\sqrt{1 - \frac{x^2}{4R^2}}\right] \qquad (0 < x < 2R). \tag{6.4}$$

This is the density of x under the null hypothesis (randomness).

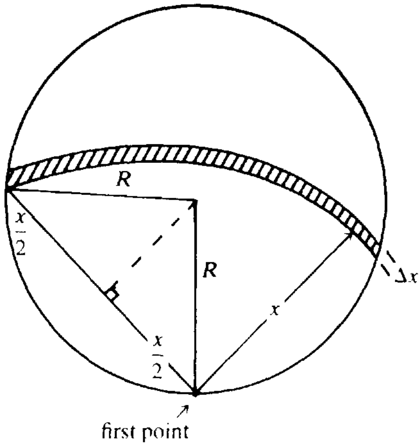


FIG. 6.2

For the alternative hypothesis we suppose there is a force of attraction (or repulsion) between the chromosomes of a pair. We shall describe this force in terms of its effect on the distribution of x, namely that the force skews the null hypothesis density, $p(x \mid H_0)$, by a factor of x^α for some power $\alpha > -2$. Thus the density of x under the alternative hypothesis, H_1, is

$$p(x \mid H_1) = Kx^\alpha p(x \mid H_0). \tag{6.5}$$

In order that $p(x \mid H_1)$ integrate to unity we must take for the value of K

$$K = \frac{(\alpha+2)\pi}{2^{\alpha+4}\beta(3/2, 1/2(\alpha+3))} = \frac{(\alpha+2)\pi}{2^{\alpha+4}\beta(3/2, (\alpha+3)/2)} \tag{6.6}$$

where $\beta(a, b)$ denotes the Beta function, $\Gamma(a)\Gamma(b)/\Gamma(a+b)$. Thus

$$p(x \mid H_1) = \frac{(\alpha+2)\pi^{\alpha+1}}{2^{\alpha+3}\beta(3/2, (\alpha+3)/2)R^2}\left[\frac{2}{\pi}\cos^{-1}\left(\frac{x}{2R}\right) - \frac{x}{2R}\sqrt{1 - \frac{x^2}{4R^2}}\right]. \tag{6.7}$$

Negative values of α shift the density to lower values of x; that is, negative values of α correspond to a force of attraction and positive values of α to a force of repulsion between the chromosomes of a pair.

Suppose n pairs of points are dropped with the same force of attraction between the members of each pair but with the members of different pairs independent of each other. Let us define

$$D = \sum_{j=1}^{n} x_j^2$$

where x_j denotes the distance between members of the jth pair. From our expressions for $p(x \mid \mathrm{H}_0)$ and $p(x \mid \mathrm{H}_1)$ we get

$$\begin{aligned} E[x^{2k} \mid \mathrm{H}_0] &= \frac{2R^{2k}}{(k+1)(k+2)}\binom{2k+1}{k+1}, \\ E[x^{2k} \mid \mathrm{H}_1] &= \frac{2^{2k}R^{2k}(\alpha+2)}{(\alpha+2k+2)} \frac{\Gamma((\alpha+3)/2+k)\Gamma(\alpha/2+3)}{\Gamma((\alpha+3)/2)\Gamma(\alpha/2+3+k)}. \end{aligned} \tag{6.8}$$

Thus we have

$$\begin{aligned} E[D \mid \mathrm{H}_0] &= nR^2, \\ \mathrm{Var}\,[D \mid \mathrm{H}_0] &= \frac{2n}{3}R^4, \\ K_3[D \mid \mathrm{H}_0] &= \frac{nR^6}{2}, \\ K_4[D \mid \mathrm{H}_0] &= \frac{nR^8}{2}, \end{aligned} \tag{6.9}$$

where K_r is the rth cumulant. Now

$$\begin{aligned} E[D \mid \mathrm{H}_1] &= 4nR^2\frac{(\alpha+2)(\alpha+3)}{(\alpha+4)(\alpha+6)}, \\ \mathrm{Var}\,[D \mid \mathrm{H}_1] &= 16nR^4\frac{(\alpha+2)(\alpha+3)(10\alpha+32)}{(\alpha+4)^2(\alpha+6)^2(\alpha+8)}, \\ K_3[D \mid \mathrm{H}_1] &= -\frac{2^9R^6n(\alpha+2)(\alpha+3)\{5\alpha^4+54\alpha^3+52\alpha^2-672\alpha-1440\}}{(\alpha+4)^3(\alpha+6)^3(\alpha+8)^2(\alpha+10)}. \end{aligned} \tag{6.10}$$

For the values of n which we are considering (23 for each karyograph), the standardized cumulants of D of order three and above are sufficiently small under both H_0 and H_1 that the distributions of D under H_0 and H_1 can be approximated by normal distributions with the means and variances given above. For example when $\alpha = 1$, we get $\gamma_1 = 0.5154n^{-1/2}$, $\gamma_2 = -0.5277n^{-1}$ for n reasonably large. Yet the normal approximation may be inadequate when $\alpha = -1$, for then $\gamma_1 = 1.7612n^{-1/2}$, $\gamma_2 = 2.9345n^{-1}$ and are not that close to zero.

One advantage of making this normal approximation is that it reduces the amount of work necessary to investigate the effect of noncircularity of the nuclear walls. Since the normal distribution is completely specified by its mean

and variance, we need only investigate the effects on the first two moments when the boundary is an ellipse and not necessarily circular.

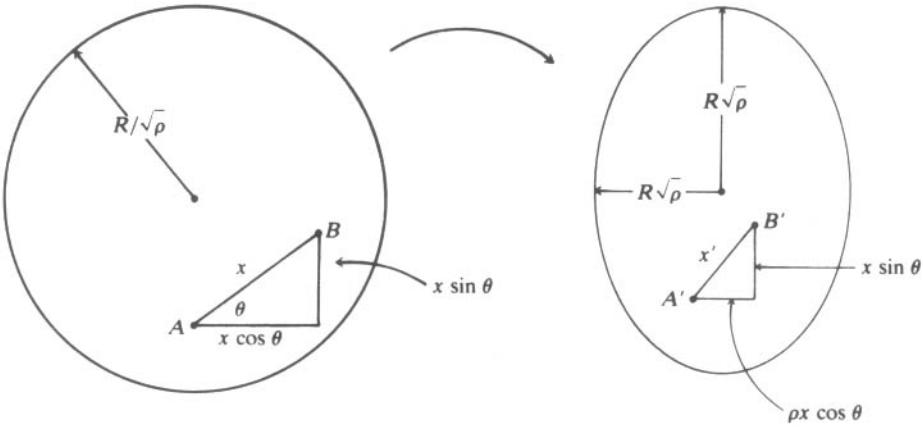


FIG. 6.3

Consider an ellipse of area πR^2 (i.e., the same area as a circle of radius R) and let ρ $(0<\rho<1)$ be the ratio of the smaller to the larger axis. This ellipse can be regarded as the result of a scaling down in one direction, by a factor ρ, of a circle of radius $R/\sqrt{\rho}$, the direction being that of the minor axis of the resultant ellipse. Any pair of points A' and B' separated by a distance x' in the ellipse will have arisen from two points A and B separated by some distance x in the circle (see Fig. 6.3). Define θ as the angle between the line AB and the direction which is scaled down when we transform the circle into the ellipse. Suppose points A and B have fallen in the circle either randomly (H_0) or with an attraction or repulsion between them (H_1). Then all values of θ between 0 and π are equally likely under both H_0 and H_1. The transformation from the circle to the ellipse leaves the component of x parallel to the major axis unchanged, while the component parallel to the minor axis is multiplied by a factor of ρ. Thus we have

$$
\begin{aligned}
E[x'^{2k}] &= E([(x\sin\theta)^2+(\rho x\cos\theta)^2]^k) \\
&= E[x^{2k}]\frac{1}{\pi}\int_0^{\pi}\sum_{j=0}^{k}\binom{k}{j}\rho^{2j}\cos^{2j}\theta\sin^{2k-2j}\theta\,d\theta.
\end{aligned}
\tag{6.11}
$$

Since $\cos^{2j}\theta\sin^{2k-2j}\theta$ is symmetric about $\pi/2$, we need integrate only from 0 to $\pi/2$.

$$
\begin{aligned}
E[x'^{2k}] &= E[x^{2k}]\frac{2}{\pi}\sum_{j=0}^{k}\binom{k}{j}\rho^{2j}\int_0^{\pi/2}\cos^{2j}\theta\sin^{2k-2j}\theta\,d\theta \\
&= E[x^{2k}]\frac{1}{\pi}\sum_{j=0}^{k}\binom{k}{j}\rho^{2j}\beta\left(\frac{2j+1}{2},\frac{2k-2j+1}{2}\right).
\end{aligned}
\tag{6.12}
$$

Thus for $k=1$ and $k=2$ we get

$$\begin{aligned} E[x'^2] &= (1/2)(1+\rho^2)E[x^2], \\ E[x'^4] &= (1/8)(3+2\rho^2+3\rho^4)E[x^4]. \end{aligned} \tag{6.13}$$

For a circle of radius R we have already obtained

$$E[x^2|\mathrm{H}_0]=R^2, \qquad E[x^2|\mathrm{H}_1]=4R^2\frac{(\alpha+2)(\alpha+3)}{(\alpha+4)(\alpha+6)}; \tag{6.14}$$

$$E[x^4|\mathrm{H}_0]=(5/3)R^4, \qquad E[x^4|\mathrm{H}_1]=16R^4\,\frac{(\alpha+2)\Gamma(\frac{1}{2}[\alpha+7])\Gamma(\frac{1}{2}\alpha+3)}{(\alpha+6)\Gamma(\frac{1}{2}[\alpha+3])\Gamma(\frac{1}{2}\alpha+5)}. \tag{6.15}$$

But the circle in which the points A and B are dropped has radius $R/\sqrt{\rho}$. If we replace R by $R/\sqrt{\rho}$ in the above expressions for $E[x^2|\mathrm{H}_0]$, $E[x^2|\mathrm{H}_1]$, $E[x^4|\mathrm{H}_0]$ and $E[x^4|\mathrm{H}_1]$ and substitute these results into our expressions for $E[x'^2]$ and $E[x'^4]$ we get

$$\begin{aligned} E[x'^2|\mathrm{H}_0] &= \left(\frac{1}{2}\right)\left[\frac{1}{\rho}+\rho\right]R^2, \\ E[x'^2|\mathrm{H}_1] &= 2\left[\frac{1}{\rho}+\rho\right]R^2\frac{(\alpha+2)(\alpha+3)}{(\alpha+4)(\alpha+6)}, \\ E[x'^4|\mathrm{H}_0] &= \frac{5}{24}\left[\frac{3}{\rho^2}+2+3\rho^2\right]R^4, \\ E[x'^4|\mathrm{H}_1] &= 2\left[\frac{3}{\rho^2}+2+3\rho^2\right]R^4\frac{\Gamma((\alpha+7)/2)\Gamma(\alpha/2+3)}{\Gamma((\alpha+3)/2)\Gamma(\alpha/2+5)}. \end{aligned} \tag{6.16}$$

Hence, if n pairs of points fall in an ellipse with major axis $R/\sqrt{\rho}$ and minor axis $\sqrt{\rho}\,R$ (that is, an ellipse whose axes have the ratio ρ $(0<\rho<1)$ and whose area is the same as that of a circle of radius R) and if

$$D'=\sum_{j=1}^{n} x_j'^2$$

where x_j' is the distance between members of the jth pair, then

$$E[D'|\mathrm{H}_0]=\frac{n}{2}\left[\frac{1}{\rho}+\rho\right]R^2, \qquad E[D'|\mathrm{H}_1]=2n\left[\frac{1}{\rho}+\rho\right]R^2\frac{(\alpha+2)(\alpha+3)}{(\alpha+4)(\alpha+6)}; \tag{6.17}$$

$$\begin{aligned} \mathrm{Var}\,[D'|\mathrm{H}_0] &= n\left[\frac{3}{8\rho^2}-\frac{1}{12}+\frac{3}{8}\rho^2\right]R^4, \\ \mathrm{Var}\,[D'|\mathrm{H}_1] &= n\left[\left(\frac{6}{\rho^2}+4+6\rho^2\right)\frac{\Gamma((\alpha+7)/2)\Gamma(\alpha/2+3)}{\Gamma((\alpha+3)/2)\Gamma(\alpha/2+5)}\right. \\ &\qquad \left. -\left(\frac{4}{\rho^2}+8+4\rho^2\right)\frac{(\alpha+2)^2(\alpha+3)^2}{(\alpha+4)^2(\alpha+6)^2}\right]R^4. \end{aligned} \tag{6.18}$$

Using normal approximations with the above means and variances for the distributions of D' under H_0 and H_1 together with data from 70 karyographs, Barton, David and Fix found that the random hypothesis is tenable for normal male cells and for abnormal male and female cells. On the other hand the random hypothesis is not tenable for normal female cells. The maximum likelihood estimator for α in that case is typically $-.4$ or $-.5$ indicating an attraction between chromosomes of homologous pairs in the normal female cell.

One question still to be investigated is whether the results of the randomization model used in the above analysis (points random in a circle) would differ very much from the results of a randomization model in which the points are random within the volume of a sphere and are then projected onto a plane, thus yielding a different kind of "random" points in a circle. This latter model would seem to better reproduce the physical condition of the three dimensional cell nucleus projected by the camera into the two dimensional karyograph. Since this latter model would tend to cluster the "random" points closer to the center of the circle, we might find that the randomness hypothesis under this latter randomization model would be tenable for normal female cells as well as the male and abnormal cells.

Random chords in the circle. Another approach to the chromosome problem is to study the lines joining the chromosomes of each pair and see whether the number of intersections of these lines within the nucleus wall could have arisen through some random mechanism. This approach was taken by David and Fix (1964). For simplicity the nucleus is taken to be circle of unit radius. Each line drawn through a homologous pair of chromosomes forms a chord of our circular nucleus. For the chromosome problem there are 23 such chords, but for generality we shall suppose we have n chords in a circle. The maximum number of intersections which these chords make within the circle is $\binom{n}{2}$, that is, every chord intersects every other chord within the circle, and $\binom{n}{2}$ is the number of pairs of chords which can be formed from a set of n chords. It is a feature of the chords drawn through the chromosomes of homologous pairs that the number of intersections within the circle averages about 80% of $\binom{n}{2}$. We want to know whether 80% of $\binom{n}{2}$ is a typical number of intersections for random chords generated by some randomization model which is consistent with the physical set-up. We shall consider six different randomization models by which random chords in a circle can be formed. For each of these models we shall derive the mean and variance of the random variable r, the total number of intersections within the circle between the n chords. Another way of measuring how frequently the random chords intersect within the circle is to count the number of intersections, w, which the n random chords make within the circle with an

additional random chord placed in the circle according to the same random mechanism as the other n chords. We shall find the mean and variance of w as well as r for each of our randomization models.

Define a random variable α_{ij} to be the indicator function of the event that the ith and jth chords intersect within the circle, that is,

$$\alpha_{ij} = \begin{cases} 1 & \text{if chords } i \text{ and } j \text{ intersect within the circle,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $E(\alpha_{ij})$ is just the probability that two particular chords intersect within the circle. We can represent w, the number of intersections with one particular chord (say chord O), as

$$w = \sum_{i=1}^{n} \alpha_{io} \tag{6.19}$$

and

$$E(w) = nE(\alpha_{io}),$$

$$E(w^2) = E\left(\sum_{i=1}^{n} \alpha_{io} \sum_{j=1}^{n} \alpha_{jo}\right) \tag{6.20}$$

$$= nE(\alpha_{io}) + n(n-1)E(\alpha_{io}\alpha_{jo}), \qquad i \neq j.$$

The other random variable of interest, r, the total number of intersections within the circle, can be written

$$r = \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \alpha_{ij} \tag{6.21}$$

so that we have

$$E(r) = \frac{1}{2} n(n-1)E(\alpha_{ij}), \tag{6.22}$$

$$E(r^2) = \frac{1}{4} E\left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \alpha_{ij} \sum_{k=1}^{n} \sum_{\substack{m=1 \\ k\neq m}}^{n} \alpha_{km}\right)$$

$$= \frac{1}{4} E\left[\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} (\alpha_{ij}\alpha_{ij} + \alpha_{ij}\alpha_{ji})\right.$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ j\neq i, i\neq k, k\neq j}}^{n} (\alpha_{ij}\alpha_{ik} + \alpha_{ij}\alpha_{ki} + \alpha_{ij}\alpha_{jk} + \alpha_{ij}\alpha_{kj})$$

$$\left. + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\substack{m=1 \\ \text{no two indices alike}}}^{n} \alpha_{ij}\alpha_{km}\right] \tag{6.23}$$

$$= \frac{1}{4}[2n(n-1)E(\alpha_{ij}) + 4n(n-1)(n-2)E(\underset{j\neq k}{\alpha_{ij}\alpha_{ik}})$$

$$+ n(n-1)(n-2)(n-3)E(\underset{i\neq j\neq k\neq m}{\alpha_{ij}\alpha_{km}})].$$

Since α_{ij} and α_{km} are independent random variables when none of the indices are the same, we have

$$\underset{i\neq j\neq k\neq m}{E(\alpha_{ij}\alpha_{km})}=[E(\alpha_{ij})]^2$$

and

$$E(r^2)=\frac{1}{4}[2n(n-1)E(\alpha_{ij})+4n(n-1)(n-2)\underset{j\neq k}{E(\alpha_{ij}\alpha_{ik})} \tag{6.24}$$
$$+n(n-1)(n-2)(n-3)(E(\alpha_{ij}))^2].$$

Thus in order to evaluate the first two moments of w and r we need only evaluate $E(\alpha_{ij})$, the probability that two random chords intersect within the circle, and $E(\alpha_{ij}\alpha_{ik})$, $j\neq k$, the probability that two random chords each intersect a third random chord within the circle.

To derive the probability that two chords intersect within the circle, we shall first derive the probability that two chords *of given lengths* intersect within the circle. The most convenient parameter by which to specify the length of a chord is half the angle subtended by the chord (θ in Fig. 6.4). Since our circle has unit radius, the arc intercepted by the chord has length 2θ.

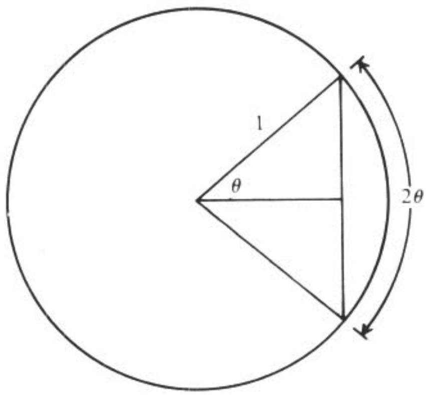


FIG. 6.4

Suppose we have two chords with half angles θ_1 and θ_2 and suppose $\theta_1>\theta_2$. Once chord 1 has been placed, chord 2 will intersect chord 1 if and only if either the counter-clockwise endpoint of chord 2 falls in shaded region A (see Fig. 6.5) or the clockwise endpoint of chord 2 falls in shaded region B. Each of these events has probability $2\theta_2/(2\pi)$. Since $\theta_2<\theta_1$, it is impossible for both events to happen simultaneously. Hence

$$P[A\cup B]=\frac{2\theta_2}{\pi}.$$

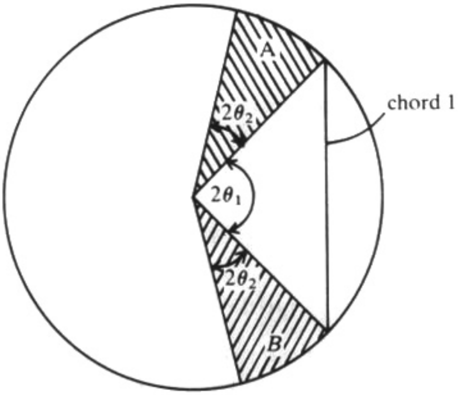


FIG. 6.5

Thus we have obtained

$$P[\text{intersection} \mid \theta_1, \theta_2] = \frac{2}{\pi} \min(\theta_1, \theta_2)$$

or, in terms of α_{12}, we have

$$E(\alpha_{12} \mid \theta_1, \theta_2) = \frac{2}{\pi} \min(\theta_1, \theta_2). \tag{6.25}$$

We can calculate $E(\alpha_{12})$ by integrating the above expression

$$\begin{aligned} E(\alpha_{12}) &= \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \min(\theta_1, \theta_2) f(\theta_1) f(\theta_2)\, d\theta_1\, d\theta_2 \\ &= \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\theta_1} \theta_2 f(\theta_2) f(\theta_1)\, d\theta_2\, d\theta_1 \end{aligned} \tag{6.26}$$

where $f(\theta)$ is the probability density function of the random variable θ (i.e. $(\Delta\theta)^{-1}$ times the probability that the random chord will have a half angle between θ and $\theta + \Delta\theta$ for infinitesimal $\Delta\theta$).

For the second moments of w and r we must calculate $E(\alpha_{12}\alpha_{13})$. When the lengths of chords 1, 2 and 3 are known (equivalently, for given θ_1, θ_2 and θ_3), the event that chords 1 and 2 intersect within the circle is independent of the event that chords 1 and 3 intersect within the circle. Thus we can write

$$E(\alpha_{12}\alpha_{13} \mid \theta_1, \theta_2, \theta_3) = E(\alpha_{12} \mid \theta_1, \theta_2) E(\alpha_{13} \mid \theta_1, \theta_3) \tag{6.27}$$

and

$$
\begin{aligned}
E(\alpha_{12}\alpha_{13}) &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} E(\alpha_{12}\,|\,\theta_1,\theta_2)E(\alpha_{13}\,|\,\theta_1,\theta_3)f(\theta_1)f(\theta_2)f(\theta_3)\,d\theta_1\,d\theta_2\,d\theta_3\\
&= \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} \min(\theta_1,\theta_2)\min(\theta_1,\theta_3)f(\theta_1)f(\theta_2)f(\theta_3)\,d\theta_1\,d\theta_2\,d\theta_3\\
&= \frac{4}{\pi^2}\int_0^{\pi/2}\Bigg[\int_0^{\theta_1}\int_0^{\theta_1}\theta_2\theta_3 f(\theta_3)f(\theta_2)\,d\theta_3\,d\theta_2\\
&\qquad + \int_{\theta_1}^{\pi/2}\int_0^{\theta_1}\theta_1\theta_3 f(\theta_3)f(\theta_2)\,d\theta_3\,d\theta_2\\
&\qquad + \int_0^{\theta_1}\int_{\theta_1}^{\pi/2}\theta_2\theta_1 f(\theta_3)f(\theta_2)\,d\theta_3\,d\theta_2 \qquad\qquad (6.28)\\
&\qquad + \int_{\theta_1}^{\pi/2}\int_{\theta_1}^{\pi/2}\theta_1^2 f(\theta_3)f(\theta_2)\,d\theta_3\,d\theta_2\Bigg]f(\theta_1)\,d\theta_1\\
&= \frac{4}{\pi^2}\int_0^{\pi/2}\Bigg[\left(\int_0^{\theta_1}\theta_2 f(\theta_2)\,d\theta_2\right)^2 + 2\theta_1\int_{\theta_1}^{\pi/2}\theta_2 f(\theta_2)\,d\theta_2\int_0^{\theta_1}\theta_3 f(\theta_3)\,d\theta_3\\
&\qquad + \theta_1^2\left(\int_{\theta_1}^{\pi/2} f(\theta_2)\,d\theta_2\right)^2\Bigg]f(\theta_1)\,d\theta_1\\
&= \frac{4}{\pi^2}\int_0^{\pi/2}\left[\int_0^{\theta_1}\theta_2 f(\theta_2)\,d\theta_2 + \theta_1\int_{\theta_1}^{\pi/2} f(\theta_2)\,d\theta_2\right]^2 f(\theta_1)\,d\theta_1.
\end{aligned}
$$

Clearly $f(\theta)$ depends on what randomization model we use to define a "random chord." We shall now consider six randomization models and derive the probability density $f(\theta)$ for each model. Then we shall be able to calculate $E(\alpha_{ij})$, $E(\alpha_{ij}\alpha_{ik})$, $j \neq k$, and the first two moments of w and r for comparison with observed values taken from karyographs. The first five models appear in David and Fix (1964) and the last model is developed by Dufour.

Model I. The simplest randomization model is to specify each chord by its endpoints, which we take to be uniformly and independently distributed over the circumference of the circle. The density of θ is, for infinitesimal $\Delta\theta$ the probability of the event that half the angle subtended by the chord is between θ and $\theta+\Delta\theta$, or, equivalently, of the event that the arc intercepted by the chord has length between 2θ and $2\theta+2\,\Delta\theta$. (See Fig. 6.6.) This event occurs if and only if the chord's second endpoint falls at a distance between 2θ and $2\theta+2\,\Delta\theta$ along the circle from the first endpoint. Since there are two such infinitesimal arcs where the second endpoint can fall (see Fig. 6.7) and each arc has length $2\,\Delta\theta$, the probability of this event is

$$P[(\theta,\theta+\Delta\theta)] = 4\,\Delta\theta/(2\pi).$$

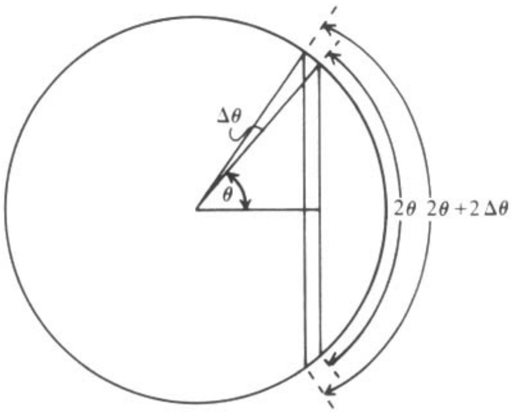


FIG. 6.6

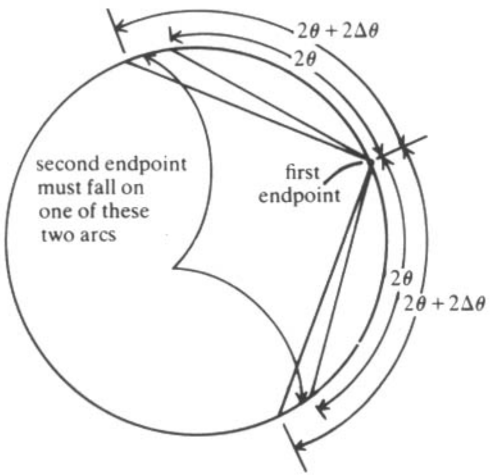


FIG. 6.7

Now $f(\theta)$ is

$$f(\theta)=\lim_{\Delta\theta\to 0}(\Delta\theta)^{-1}P[(\theta,\theta+\Delta\theta)].$$

Thus

$$f(\theta)=\frac{2}{\pi}\qquad(0\leqq\theta\leqq\pi/2). \tag{6.29}$$

Model II. Any chord of a circle (except a diameter) is completely specified by the length and direction of the line segment from the center of the circle to the midpoint of the chord. For this randomization model let the length, p, of this line segment be uniformly distributed on $[0, 1]$ and let the angle ϕ which this line

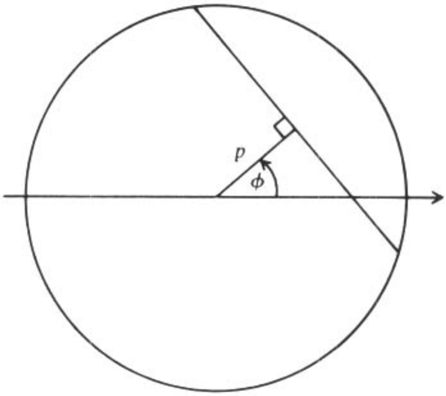


FIG. 6.8a

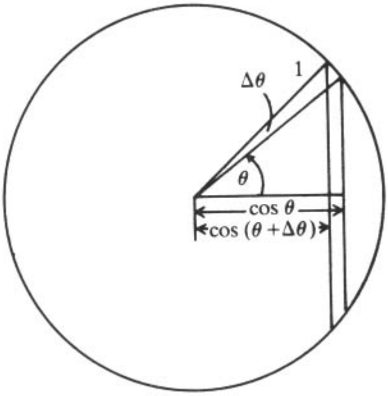


FIG. 6.8b

segment makes with some axis be uniformly distributed on $[0, 2\pi]$. (See Fig. 6.8a.) A chord will intercept an arc of half angle between θ and $\theta+\Delta\theta$ if and only if p falls between $\cos(\theta+\Delta\theta)$ and $\cos\theta$ (see Fig. 6.8b). The probability of this event is $\cos(\theta)-\cos(\theta+\Delta\theta)$ which is approximately $\sin(\theta)\,\Delta\theta$ for very small $\Delta\theta$. Thus in this case the density of θ is

$$f(\theta)=\sin\theta \qquad (0\leqq\theta\leqq\pi/2). \tag{6.30}$$

Model III. For this model suppose that a point is dropped at random in the circle and the chord is drawn which has this point as its midpoint, that is, the chord is drawn perpendicular to the line joining the random point with the center of the circle. (If the perpendicular to the chord from the center of the circle has length p and makes an angle ϕ with some prescribed axis, then the joint density function of p and ϕ is $(1/\pi)p\,dp\,d\phi$, $0\leqq p\leqq 1$, $0\leqq\phi\leqq 2\pi$.) Then a chord will intercept an arc of half angle between θ and $\theta+\Delta\theta$ if and only if its midpoint falls in the annulus between the circles of radius $\cos\theta$ and $\cos(\theta+\Delta\theta)$ (see Fig. 6.9). The probability of this event is $\pi[\cos^2\theta-\cos^2(\theta+\Delta\theta)]/\pi$ which, for infinitesimal $\Delta\theta$, is $2\cos\theta\sin\theta\,\Delta\theta$. Thus

$$f(\theta)=2\cos\theta\sin\theta \qquad (0\leqq\theta\leqq\pi/2). \tag{6.31}$$

Model IV. Let us now define a random chord by dropping a point at random in the circle and drawing a chord through that point in a randomly chosen direction. Suppose the random point falls at a distance between x and $x+\Delta x$ from the center of the circle. This event has a probability equal to the area of an annulus of radius x and width Δx, divided by the total area of the circle, that is, $2\pi x\,\Delta x/\pi$ or $2x\,\Delta x$. Thus the density of x is $2x$ for $0\leqq x\leqq 1$. Given that the random point falls at a distance x from the center of the circle, the random chord intercepts an arc of half angle between θ and $\theta+\Delta\theta$ if and only if the random chord falls in one of the shaded regions A and B shown in Fig. 6.10, that is, if and only if the random direction falls between two chords which intercept arcs of 2θ and $2\theta+2\,\Delta\theta$ and which cross at a distance x from the center of the circle.

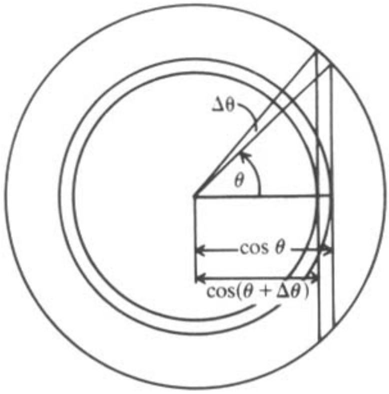


FIG. 6.9

The region B in Fig. 6.10 is just a reflection of region A. Therefore the total angular measure of the region $A \cup B$ about the random point is just twice the angle $\Delta\beta$ of the region A. Thus the probability which we seek is just $2\,\Delta\beta/\pi$. The two arcs s_1 and s_2 in Fig. 6.10 must satisfy

(6.32) $$s_1 - s_2 = 2\,\Delta\theta.$$

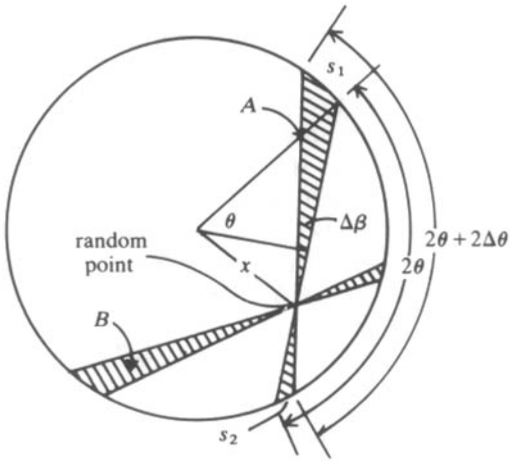


FIG. 6.10

The relation between s_1, s_2 and $\Delta\beta$ is evident from a more detailed version of Fig. 6.10. First it should be noted that the shorter chord makes an angle θ with the circle. For small $\Delta\theta$ the longer chord is almost coincident with the shorter chord and it makes an angle approximately θ with the circle. Thus the width at the top of the region A is approximately $s_1 \sin\theta$ when $\Delta\theta$ is very small. This width can also be written as $\Delta\beta$ times the distance from the random point to the

top of the region A. (See Fig. 6.11.) Thus we have approximately (exactly for infinitesimal $\Delta\theta$)

$$\Delta\beta(\sin\theta+\sqrt{x^2-\cos^2\theta})=s_1\sin\theta \tag{6.33}$$

and similarly

$$\Delta\beta(\sin\theta-\sqrt{x^2-\cos^2\theta})=s_2\sin\theta.$$

Solving for s_1 and s_2 and substituting into $s_1-s_2=2\,\Delta\theta$ we get

$$\Delta\beta\left[\frac{\sin\theta+\sqrt{x^2-\cos^2\theta}}{\sin\theta}-\frac{\sin\theta-\sqrt{x^2-\cos^2\theta}}{\sin\theta}\right]=2\,\Delta\theta \tag{6.34}$$

which yields

$$\Delta\beta=\frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-\cos^2\theta}}.$$

Given that the random point falls at distance x from the center of the circle, the probability that the random chord intercepts an arc of half angle between θ and $\theta+\Delta\theta$ is $2\,\Delta\beta/\pi$ or

$$P[(\theta,\theta+\Delta\theta)\,|\,x]=\frac{2\sin\theta\,\Delta\theta}{\pi\sqrt{x^2-\cos^2\theta}}. \tag{6.35}$$

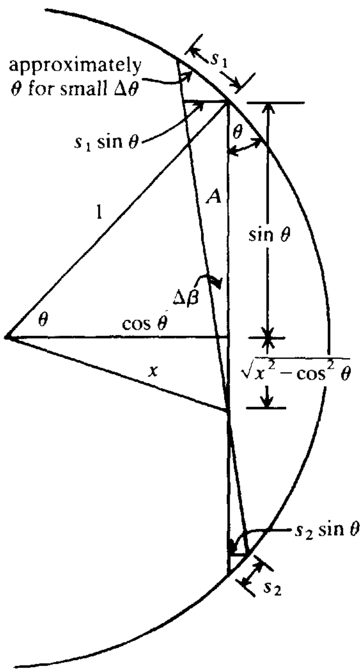


FIG. 6.11

The unconditional probability is obtained by integrating the above expression over the density of x. We have shown that the density of x is $2x$ for $0 \leqq x \leqq 1$. Note that we need integrate only from $x = \cos\theta$ to $x = 1$ since it is impossible for a chord to intercept an arc of half angle θ if any point on the chord is closer than $\cos\theta$ to the center of the circle. Thus

$$\begin{aligned} P[\theta, \theta+\Delta\theta] &= \int_{\cos\theta}^{1} \frac{2\sin\theta\,\Delta\theta}{\pi\sqrt{x^2-\cos^2\theta}}\,2x\,dx \\ &= \frac{4}{\pi}\sin\theta\,\Delta\theta\left[\sqrt{x^2-\cos^2\theta}\right]_{x=\cos\theta}^{x=1} \\ &= \frac{4}{\pi}\sin^2\theta\,\Delta\theta. \end{aligned}$$

Hence for infinitesimal $\Delta\theta$ we have

$$f(\theta) = \frac{4}{\pi}\sin^2\theta \qquad \left(0 \leqq \theta \leqq \frac{\pi}{2}\right). \tag{6.36}$$

Model V. For the fifth randomization model let us generate our random chord by dropping two points randomly within the circle and drawing a chord through both points. The derivation of $f(\theta)$ for this model is very similar to that for the previous model. The density of the distance x from the first random point to the center of the circle is, as before, $2x$. Given that the first point is at distance x from the center of the circle, the second random point must fall in the region $A \cup B$ as defined for the previous model in order that the chord joining the two points intercepts an arc of half angle between θ and $\theta+\Delta\theta$. The difference between this and the previous model is that there our probability was the total angular measure of $A \cup B$ divided by 2π, whereas here our probability is the total area of $A \cup B$ over the area of the circle, π. The area of a wedge shaped region of radius R and angle γ is $\frac{1}{2}R^2\gamma$. (See Fig. 6.12.) Thus the area of the region A is (except for terms of order $(\Delta\theta)^2$)

$$\begin{aligned} \text{Area}(A) &= \frac{1}{2}(\sin\theta+\sqrt{x^2-\cos^2\theta})^2\,\Delta\beta + \frac{1}{2}(\sin\theta-\sqrt{x^2-\cos^2\theta})^2\,\Delta\beta \\ &= (x^2-\cos^2\theta+\sin^2\theta)\,\Delta\beta. \end{aligned} \tag{6.37}$$

The probability that the second random point falls in the region $A \cup B$ is

$$\frac{1}{\pi}\,\text{Area}(A \cup B) = \frac{2}{\pi}\,\text{Area}(A) = \frac{2}{\pi}(x^2-\cos^2\theta+\sin^2\theta)\,\Delta\beta. \tag{6.38}$$

As in the previous model we have

$$\Delta\beta = \frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-\cos^2\theta}}.$$

Hence, given that the first random point falls at distance x from the center of the

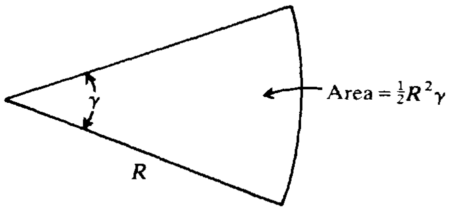


FIG. 6.12

circle, the probability that the chord going through the two random points intercepts an arc of half angle between θ and $\theta+\Delta\theta$ is

$$P[(\theta, \theta+\Delta\theta)|x] = \frac{2}{\pi}(x^2 - \cos^2\theta + \sin^2\theta)\frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-\cos^2\theta}}$$

and, as in the previous model

$$\begin{aligned}
P[(\theta, \theta+\Delta\theta)] &= \int_{\cos\theta}^{1} \frac{2}{\pi}(x^2-\cos^2\theta+\sin^2\theta)\frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-\cos^2\theta}}2x\,dx \\
&= \frac{2}{\pi}\sin\theta\,\Delta\theta\int_{\cos\theta}^{1}(x^2-\cos^2\theta)^{1/2}2x\,dx \\
&\quad+\frac{2}{\pi}\sin^3\theta\,\Delta\theta\int_{\cos\theta}^{1}\frac{2x\,dx}{(x^2-\cos^2\theta)^{1/2}} \\
&= \frac{2}{\pi}\sin\theta\,\Delta\theta\left[\frac{2}{3}(x^2-\cos^2\theta)^{3/2}\right]_{x=\cos\theta}^{x=1} \\
&\quad+\frac{2}{\pi}\sin^3\theta\,\Delta\theta[2(x^2-\cos^2\theta)^{1/2}]_{x=\cos\theta}^{x=1} \\
&= \frac{4}{3\pi}\sin^4\theta\,\Delta\theta+\frac{4}{\pi}\sin^4\theta\,\Delta\theta \\
&= \frac{16}{3\pi}\sin^4\theta\,\Delta\theta.
\end{aligned} \tag{6.39}$$

Dividing by $\Delta\theta$ and letting $\Delta\theta$ become infinitesimal we get

$$f(\theta) = \frac{16}{3\pi}\sin^4\theta \qquad (0 \leqq \theta \leqq \pi/2). \tag{6.40}$$

Model VI. The randomization model which is perhaps most applicable to the distribution of chromosomes in a cell nucleus is the following. Consider the nucleus to be a sphere of unit radius, let two points be placed at random inside the sphere and project these two points onto a plane passing through the center of the sphere. The sphere projects onto the plane as a circle of unit radius and the two projected points in this circle determine a random chord as in the

previous model. This model is an attempt to take into account the fact that the camera projects the three-dimensional cell nucleus onto a two dimensional picture. Our method of obtaining $f(\theta)$ for this model will be very similar to that used for the previous two models. The first random point in the sphere must fall in a cylindrical shell of radius x, thickness Δx and length $2\sqrt{1-x^2}$ (see Fig. 6.13) in order that its projection lie at a distance between x and $x+\Delta x$ from the center of the circle. The probability of this event is the ratio of the volume of the cylindrical shell to the volume of the sphere, namely

$$
\begin{aligned}
P[(x, x+\Delta x)] &= \frac{(2\pi x\ \Delta x)(2\sqrt{1-x^2})}{(4/3)\pi} \\
&= 3x\ \Delta x\sqrt{1-x^2}.
\end{aligned}
\tag{6.41}
$$

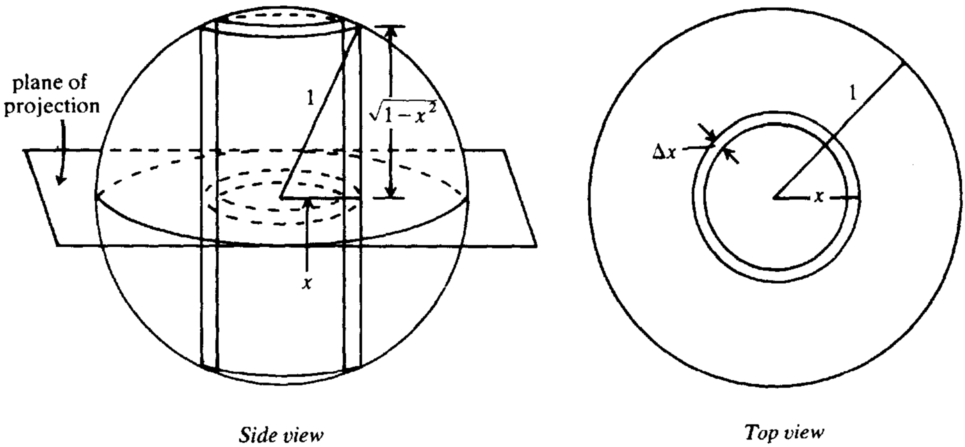


FIG. 6.13

Thus dividing by Δx and letting Δx become infinitesimal, we find that the probability density of x, the distance from the projection of the first random point to the center of the circle, is $3x\sqrt{1-x^2}$. Now suppose the first projected point falls at distance x from the center of the circle. Our random chord will intercept an arc of half angle between θ and $\theta+\Delta\theta$ if and only if the projection of the second random point falls in the region $A \cup B$ described for Model IV. Thus the probability that we seek is just the volume of the sphere directly above and below the region $A \cup B$ divided by the total volume of the sphere, $\frac{4}{3}\pi$. By symmetry the volume above and below $A \cup B$ is twice the volume above and below A. We shall get the volume above and below the region A by considering elements of area in A which are at a distance between η and $\eta+\Delta\eta$ from the center of the circle (see Fig. 6.14a). For each value of η there are two arcs whose lengths we shall call Δt_1 and Δt_2 such that these arcs comprise the points in A at distance η from the center of the circle. Similarly, the two thin strips of length Δt_1 and Δt_2 in Fig. 6.14a contain all the points in A at distance between η and

$\eta+\Delta\eta$ from the center of the circle. The volume in the sphere above and below these strips is just the area of the two strips times twice the height of the surface of the sphere above the strips, i.e., $[\Delta t_1+\Delta t_2]\,\Delta\eta(2\sqrt{1-\eta^2})$. We must now obtain expressions for Δt_1 and Δt_2 in terms of η. For brevity let $p=\cos\theta$, the distance from the center of the circle to the shorter of the two chords making up the boundary of the region A. As before, let $\Delta\beta$ denote the angle between these two chords. Let us consider the arc Δw_1 as shown in Fig. 6.14b.

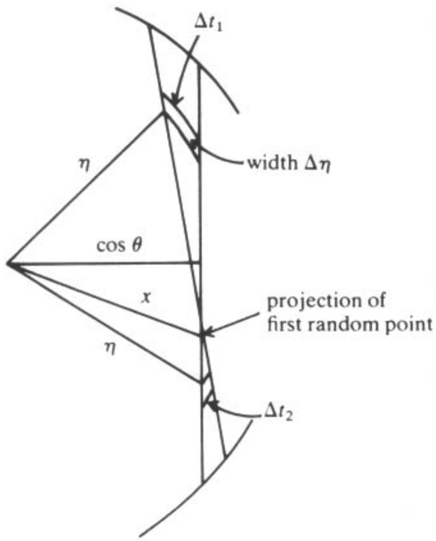


FIG. 6.14a

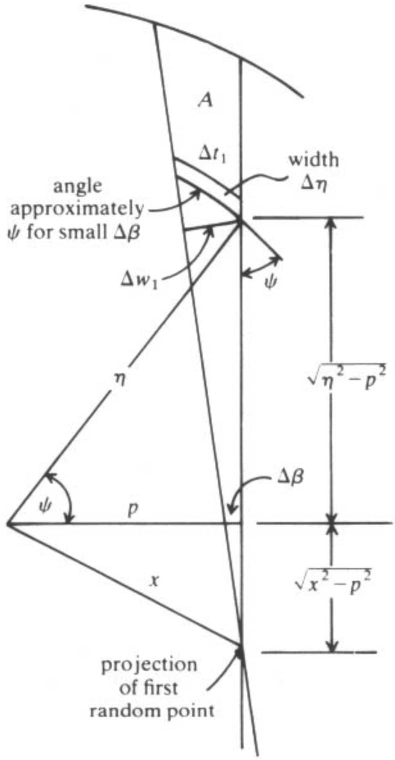


FIG. 6.14b

The length of Δw_1 is

$$\Delta w_1=(\sqrt{x^2-p^2}+\sqrt{\eta^2-p^2})\,\Delta\beta. \tag{6.42}$$

For every small Δt_1 and Δw_1 these arcs can be assumed linear and by the same argument as used in Model IV to show that the width at the rop of the region A is $s_1\sin\theta$ we see that

$$\Delta w_1=\Delta t_1\sin\psi$$

where ψ is defined in Fig. 6.14b ($\psi=\cos^{-1}(p/\eta)$). Equating our two expressions

for Δw_1 and solving for Δt_1 we get

$$\Delta t_1 = \frac{1}{\sin\psi}(\sqrt{x^2-p^2}+\sqrt{\eta^2-p^2})\,\Delta\beta \tag{6.43}$$
$$= \frac{\eta}{\sqrt{\eta^2-p^2}}(\sqrt{x^2-p^2}+\sqrt{\eta^2-p^2})\,\Delta\beta.$$

In the same manner we obtain

$$\Delta t_2 = \begin{cases} \dfrac{\eta}{\sqrt{\eta^2-p^2}}(\sqrt{\eta^2-p^2}-\sqrt{x^2-p^2})\,\Delta\beta, & \eta > x, \\ \dfrac{\eta}{\sqrt{\eta^2-p^2}}(\sqrt{x^2-p^2}-\sqrt{\eta^2-p^2})\,\Delta\beta, & \eta \leqq x, \end{cases} \tag{6.44}$$

so that

$$\Delta t_1 + \Delta t_2 = \begin{cases} 2\eta\,\Delta\beta, & \eta > x, \\ \dfrac{2\eta\sqrt{x^2-p^2}}{\sqrt{\eta^2-p^2}}\,\Delta\beta, & \eta \leqq x. \end{cases}$$

The volume of that portion of the sphere which is projected onto the two strips Δt_1 and Δt_2 is

$$\text{Volume projected on } \Delta t_1 \text{ and } \Delta t_2 = 2[\Delta t_1+\Delta t_2]\sqrt{1-\eta^2}\,\Delta\eta$$
$$= \begin{cases} 4\,\Delta\beta\eta\sqrt{1-\eta^2}\,\Delta\eta, & \eta > x, \\ 4\,\Delta\beta\dfrac{\eta\sqrt{x^2-p^2}\sqrt{1-\eta^2}}{\sqrt{\eta^2-p^2}}\Delta\eta, & \eta \leqq x. \end{cases} \tag{6.45}$$

If we now take $\Delta\eta$ to be infinitesimal and integrate over all values of η between p and 1 we get the total volume of the portion of the sphere which is projected onto the region A (except for terms of order $(\Delta\beta)^2$)

Volume projected onto A

$$= 4\,\Delta\beta\left[\sqrt{x^2-p^2}\int_p^x \frac{\eta\sqrt{1-\eta^2}}{\sqrt{\eta^2-p^2}}\,d\eta + \int_x^1 \eta\sqrt{1-\eta^2}\,d\eta\right] \tag{6.46}$$
$$= 2\,\Delta\beta\left[(x^2-p^2)\sqrt{1-x^2}+(1-p^2)\sqrt{x^2-p^2}\tan^{-1}\sqrt{\frac{x^2-p^2}{1-x^2}} + \frac{2}{3}(1-x^2)^{3/2}\right].$$

As in the previous two models we have

$$\Delta\beta = \frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-\cos^2\theta}} = \frac{\sin\theta\,\Delta\theta}{\sqrt{x^2-p^2}}.$$

Thus

Volume projected onto A

$$= 2 \sin \theta \, \Delta\theta \left[\sqrt{(1-x^2)(x^2-p^2)} + (1-p^2) \tan^{-1} \sqrt{\frac{x^2-p^2}{1-x^2}} + \frac{2}{3} \frac{(1-x^2)^{3/2}}{\sqrt{x^2-p^2}} \right].$$

Given the distance x of the first projected point from the center of the circle, the probability that our random chord intercepts an arc of half angle between θ and $\theta + \Delta\theta$ is

$$P[(\theta, \theta + \Delta\theta) \mid x] = \frac{\text{Volume projected onto } A \cup B}{\text{Volume of sphere}}$$

$$= \frac{2(\text{Volume projected onto } A)}{(4/3)\pi}$$

(6.47)

$$= \frac{3}{\pi} \sin \theta \, \Delta\theta \left[\sqrt{(1-x^2)(x^2-p^2)} + (1-p^2) \tan^{-1} \sqrt{\frac{x^2-p^2}{1-x^2}} + \frac{2}{3} \frac{(1-x^2)^{3/2}}{\sqrt{x^2-p^2}} \right].$$

As in the previous two models we now integrate over the density of x from $\cos \theta$ (which we are calling p) to 1. Remember that in this case the density of x is $3x\sqrt{1-x^2}$.

$$P[(\theta, \theta + \Delta\theta)] = \frac{9}{\pi} \sin \theta \, \Delta\theta \int_p^1 \left[\sqrt{(1-x^2)(x^2-p^2)} + (1-p^2) \tan^{-1} \sqrt{\frac{x^2-p^2}{1-x^2}} + \frac{2}{3} \frac{(1-x^2)^{3/2}}{\sqrt{x^2-p^2}} \right] x\sqrt{1-x^2} \, dx$$

(6.48)

$$= \frac{9}{\pi} \sin \theta \, \Delta\theta \left[\frac{2}{15}(1-p^2)^{5/2} + \frac{2}{9}(1-p^2)^{5/2} + \frac{16}{45}(1-p^2)^{5/2} \right]$$

$$= \frac{32}{5\pi} \sin \theta \, \Delta\theta (1-p^2)^{5/2}$$

$$= \frac{32}{5\pi} \sin^6 \theta \, \Delta\theta \qquad (\text{since } p = \cos \theta).$$

Hence for our final randomization model

$$f(\theta) = \frac{32}{5\pi} \sin^6 \theta \qquad (0 \leqq \theta \leqq \pi/2).$$

We are indebted to Dr. William A. Visscher[2] for the following way of developing $f(\theta)$ for Models V and VI.

[2] Dr. Visscher of Los Alamos Laboratories attended the Lecture Series at the University of Nevada and submitted the derivation above by letter very shortly after the lectures terminated.

Take 2 points at random uniformly in an n-dimensional sphere of unit radius. Project the straight line connecting them onto a circular projection of the sphere. What is the distribution of lengths of chords thus found?

DEFINITIONS. (ρ_1, φ_1), (ρ_2, φ_2) are projections onto the circle of the two random points in n-dimensional spheres. One can read from Fig. 6.15 that

$$(6.49) \qquad |\varphi_2 - \varphi_1| = \varphi = \left|\arccos \frac{\cos\theta}{\rho_1} \pm \arccos \frac{\cos\theta}{\rho_2}\right|;$$

i.e. for each value of (θ, ρ_1, ρ_2) there are either two values of φ (see dashed line) or none (if $\rho_2 = \min(\rho_1, \rho_2) < \cos\theta$). The density function for $\cos\theta$ is then

$$(6.50) \qquad f(\cos\theta) = E\{\delta(\cos\theta - C(\varphi, \rho_1, \rho_2))\}$$

where C is the solution of (6.49) for $\cos\theta$, and E means averaging the 2 value points over the n-sphere volume. By construction we have $\int_0^1 f(\cos\theta)\, d\cos\theta = 1$. The volume integral in n-dimensions is

$$(6.51) \qquad \int_0^1 \rho\, d\rho \int_0^{2\pi} d\varphi \int_{x_3^2+\cdots+x_n^2<1-\rho^2} dx_3 \cdots dx_n = v_{n-2} \int_0^1 \rho\, d\rho \int_0^{2\pi} d\varphi (1-\rho^2)^{(n-1)/2}$$

where the volume of a sphere in n-dimensions is

$$V_n = v_n R^n;$$

$$v_0 = 1, \quad v_1 = 2, \quad v_2 = \pi, \quad v_3 = \frac{4\pi}{3}, \quad v_4 = \pi^2, \quad v_5 = \frac{8\pi^2}{15}, \quad v_6 = \frac{\pi^3}{3}, \quad v_7 = \frac{16\pi^3}{105}, \cdots.$$

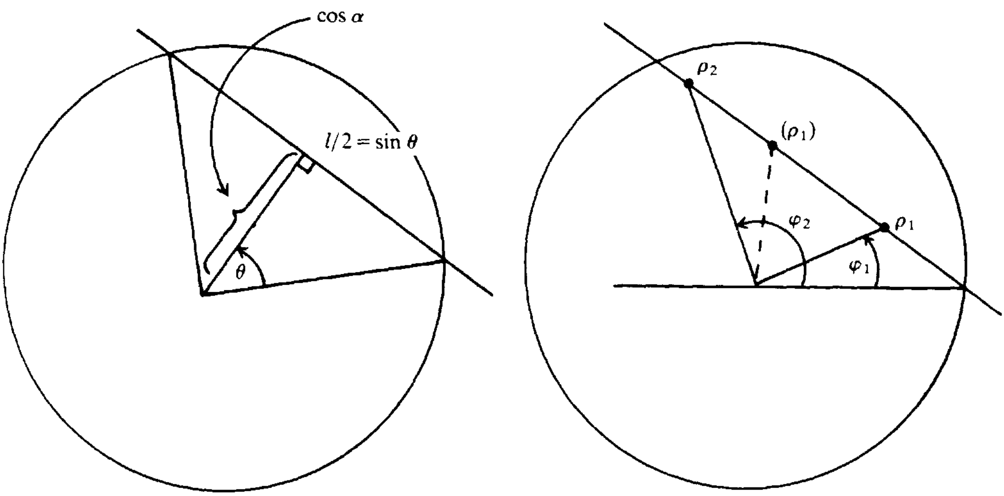


FIG. 6.15

Equation (6.50) then becomes

$$(6.52)\quad \begin{aligned} f(\cos\theta) = \frac{v_{n-2}^2}{v_n^2} \int_0^1 \rho_1\, d\rho_1 (1-\rho_1^2)^{(n-2)/2} \int_0^1 \rho_2\, d\rho_2 (1-\rho_2^2)^{(n-2)/2} \\ \cdot 4\pi \int_0^\pi d\varphi\, \delta(\cos\theta - C(\varphi, \rho_1, \rho_2)), \end{aligned}$$

and the integral over φ can be done, using

$$(6.53)\quad \delta(\cos\theta - C(\varphi,\rho_1,\rho_2)) = \delta\left(\varphi - \left|\arc\cos\frac{\cos\theta}{\rho_1} \pm \arc\cos\frac{\cos\theta}{\rho_2}\right|\right)\left|\frac{\delta\varphi}{\delta\cos\theta}\right|,$$

and

$$(6.54)\quad \left|\frac{\delta\varphi}{\delta\cos\theta}\right| = \frac{2}{\sqrt{\rho_<^2 - \cos^2\theta}},$$

after the effect of 2 values of φ for each θ is accounted for. With (6.54) inserted into (6.52), and the ρ_1, ρ_2 integral split into $\rho_2<\rho_1$ and $\rho_2>\rho_1$ parts which give equal contributions, the integration is straightforward and gives

$$(6.55)\quad \begin{aligned} f(\cos\theta) &= \frac{8\pi v_{n-2}^2}{v_n^2} \int_{\cos\theta}^1 \rho_1\, d\rho_1 (1-\rho_1^2)^{(n-2)/2} \int_{\cos\theta}^1 \rho_2\, d\rho_2 (1-\rho_2^2)^{(n-2)/2} \frac{2}{\sqrt{\rho_<^2 - \cos^2\theta}} \\ &= \frac{8\pi}{n} \frac{v_{n-2}^2}{v_n^2} \frac{(n-1)!2^n}{1\cdot 3\cdot 5\cdots(2n-1)} \sin^{2n-1}. \end{aligned}$$

This gives, for $n=2$

$$f(\cos\theta) = \frac{16}{2\pi}\sin^3\theta$$

and for $n=3$

$$f(\cos\theta) = \frac{32}{5\pi}\sin^5\theta.$$

We now use the above expressions for $f(\theta)$ to calculate $E(w)$, Var (w), $E(r)$ and Var (r) for each of our randomization models. For example, for Model I we have

$$\begin{aligned} E(\alpha_{12}) &= \frac{4}{\pi}\int_0^{\pi/2}\int_0^{\theta_1} \theta_2 f(\theta_2) f(\theta_1)\, d\theta_2\, d\theta_1 \\ &= \frac{4}{\pi}\int_0^{\pi/2}\int_0^{\theta_1} \theta_2 \left(\frac{2}{\pi}\right)\left(\frac{2}{\pi}\right) d\theta_2\, d\theta_1 \\ &= \frac{16}{\pi^3}\int_0^{\pi/2} \frac{\theta_1^2}{2}\, d\theta_1 = \frac{16}{\pi^3}\cdot\frac{1}{6}\left(\frac{\pi}{2}\right)^3 \\ &= \frac{1}{3}. \end{aligned}$$

Thus the expected number of intersections within the circle between n random chords and an additional random chord is

$$E(w)=nE(\alpha_{12})=\frac{1}{3}n$$

and the expected total number of intersections between the n random chords for Model I is

$$E(r)=\frac{1}{2}n(n-1)E(\alpha_{12})=\frac{1}{6}n(n-1).$$

To get the second moments of w and r for Model I we calculate

$$\begin{aligned}E(\alpha_{12}\alpha_{13})&=\frac{4}{\pi^2}\int_0^{\pi/2}\left[\int_0^{\theta_1}\theta_2\left(\frac{2}{\pi}\right)d\theta_2+\theta_1\int_{\theta_1}^{\pi/2}\left(\frac{2}{\pi}\right)d\theta_2\right]^2\left(\frac{2}{\pi}\right)d\theta_1\\&=\frac{32}{\pi^3}\int_0^{\pi/2}\left[\frac{\theta_1^2}{2}+\theta_1\left(\frac{\pi}{2}-\theta_1\right)\right]^2 d\theta_1\\&=\frac{2}{15}.\end{aligned}$$

Hence substituting into our equation

$$E(w^2)=nE(\alpha_{io})+n(n-1)E(\alpha_{io}\alpha_{jo}), \qquad i\neq j,$$

we get

$$E(w^2)=\frac{2}{15}n^2+\frac{1}{5}n$$

and

$$\operatorname{Var}(w)=\frac{n(n+9)}{45}.$$

Our expression for the second moment of r

$$\begin{aligned}E(r^2)=\frac{1}{4}[2n(n-1)E(\alpha_{12})+4n(n-1)(n-2)E(\alpha_{12}\alpha_{13})\\+n(n-1)(n-2)(n-3)(E(\alpha_{12}))^2]\end{aligned}$$

yields

$$E(r^2)=\frac{1}{180}n(n-1)(5n^2-n+12)$$

and

$$\operatorname{Var}(r)=\frac{n(n-1)(n+3)}{45}.$$

The calculation of the first two moments of w and r for the other five randomization models can be carried out in exactly the same manner as for Model I above. The results of these calculations are summarized in Table 4.

The column of primary interest in this table is the column containing the values of $E(\alpha_{ij})$, the probability that two particular chords intersect within the circle. As was remarked earlier, it is a feature of chromosomes that the chords joining homologous pairs intersect about 80% of the time, that is, $E(\alpha_{ij})$ for the chords joining homologous pairs of chromosomes is about .8. No randomization model we have looked at gives such a high value. Model VI is closest with the value of .728. We could easily devise a model, for example with $f(\theta)=(512/(63\pi))\sin^{10}\theta$, which would give a value of approximately .8 for $E(\alpha_{ij})$, but it is doubtful whether we could justify such a model on the basis of the physical set-up. The pairs of chromosomes seem to be balanced with respect to the center of the circle, thus causing the chords drawn through the pairs to intersect more often than if the members of each pair were random with respect to each other and the center of the circle.

More results on random chords. We can derive additional results concerning the intersections of random chords for several of the randomization models which we considered above. For Model I, for example, where the random chord is determined by two points independently and randomly placed on the circumference, a simpler derivation of $E(\alpha_{12})$ was given by W. A. Whitworth in *Choice and Chance* (1897). Not only is Whitworth's method simpler, but it applies to random chords in any convex region, and it yields the probability mass function of w, the number of intersections between n random chords and an additional random chord. For this approach we shall normalize the circumference of our convex region to be unity.

First we shall derive $E(\alpha_{12})$, the probability that two random chords intersect within the convex region. Let the first chord intercept an arc of length x. Then x is uniformly distributed on $[0, 1]$. The two chords intersect if one of the endpoints of the second chord falls on the arc of length x and the other falls on the remainder of the circle (of length $1-x$). There are two ways this event can happen, namely, the first endpoint can fall on the arc of length x and the second outside, or vice versa. Thus we have

$$E(\alpha_{12}|x)=2x(1-x)$$

and

$$E(\alpha_{12})=\int_0^1 2x(1-x)\,dx=\frac{1}{3}$$

which agrees with the result for the circle by our previous method. It is interesting to note that for this randomization model $E(\alpha_{12})$ is independent of the shape of the convex region.

$E(\alpha_{12}\alpha_{13})$ is equally easy to derive by Whitworth's method. Given that chord 1 intercepts an arc of length x, the event that chord 2 intersects chord 1 is

TABLE 4

Model	Method of determining random chord	$f(\theta)$	$E(\alpha_{ij})$	$E(w)$	$E(r)$	$E(\alpha_{ij}\alpha_{ij})$ $j \neq k$	Var (w)	Var (r)
I	Two random points on circumference	$\frac{2}{\pi}$	$\frac{1}{3}$ (.333)	$\frac{1}{3}n$	$\frac{1}{6}n(n-1)$	$\frac{2}{15}$	$\frac{n^2}{45}+\frac{n}{5}$	$\frac{n(n-1)(n+3)}{45}$
II	Uniform distance to origin and uniform angle with axes $\left(\frac{1}{2\pi}\,dp\,d\varphi\right)$	$\sin\theta$	$\frac{1}{2}$ (.500)	$\frac{1}{2}n$	$\frac{1}{4}n(n-1)$	$\frac{8}{3\pi^2}$	$\left(\frac{8}{3\pi^2}-\frac{1}{4}\right)n^2 +\left(\frac{1}{2}-\frac{8}{3\pi^2}\right)n$	$n(n-1)\left[\left(\frac{8}{3\pi^2}-\frac{1}{4}\right)n +\left(\frac{5}{8}-\frac{16}{3\pi^2}\right)\right]$
III	Midpoint random in the circle $\left(\frac{1}{\pi}p\,dp\,d\varphi\right)$	$2\cos\theta\sin\theta$	$\frac{3}{8}$ (.375)	$\frac{3}{8}n$	$\frac{3}{16}n(n-1)$	$\frac{3}{16}-\frac{1}{3\pi^2}$	$\left(\frac{3}{64}-\frac{1}{3\pi^2}\right)n^2 +\left(\frac{3}{16}+\frac{1}{3\pi^2}\right)n$	$n(n-1)\left[\left(\frac{3}{64}-\frac{1}{3\pi^2}\right)n +\left(\frac{3}{128}+\frac{2}{3\pi^2}\right)\right]$
IV	One random point in the circle and a random direction	$\frac{4}{\pi}\sin^2\theta$	$\frac{1}{3}+\frac{5}{2\pi^2}$ (.587)	$\left(\frac{1}{3}+\frac{5}{2\pi^2}\right)n$	$\left(\frac{1}{6}+\frac{5}{4\pi^2}\right)n(n-1)$	$\frac{2}{15}+\frac{1}{\pi^2}+\frac{49}{4\pi^4}$	$\left(\frac{1}{45}-\frac{2}{3\pi^2}+\frac{6}{\pi^4}\right)n^2 +\left(\frac{1}{5}+\frac{3}{2\pi^2}-\frac{49}{4\pi^4}\right)n$	$n(n-1) \cdot\left[\left(\frac{1}{45}-\frac{2}{3\pi^2}+\frac{6}{\pi^4}\right)n +\left(\frac{1}{15}+\frac{7}{4\pi^2}-\frac{121}{8\pi^4}\right)\right]$

TABLE 4 (cont.)

V	Two random points in the circle	$\frac{16}{3\pi}\sin^4\theta$	$\frac{1}{3}+\frac{245}{72\pi^2}$ (.678)	$\left(\frac{1}{3}+\frac{245}{72\pi^2}\right)n$	$\left(\frac{1}{6}+\frac{245}{144\pi^2}\right)\cdot n(n-1)$	$\frac{2}{15}+\frac{155}{108\pi^2}+\frac{97097}{5184\pi^4}$	$\left(\frac{1}{45}-\frac{5}{6\pi^2}+\frac{2317}{324\pi^4}\right)n^2+\left(\frac{1}{5}+\frac{425}{216\pi^2}-\frac{97097}{5184\pi^4}\right)n$	$n(n-1)\cdot\left[\left(\frac{1}{45}-\frac{5}{6\pi^2}+\frac{2317}{324\pi^4}\right)n+\left(\frac{1}{15}+\frac{965}{432\pi^2}-\frac{208313}{3^4 2^7\pi^4}\right)\right]$
VI	Two random points in a sphere projected onto the plane	$\frac{32}{5\pi}\sin^6\theta$	$\frac{1}{3}+\frac{7007}{1800\pi^2}$ (.728)	$\left(\frac{1}{3}+\frac{7007}{1800\pi^2}\right)n$	$\left(\frac{1}{6}+\frac{7007}{3600\pi^2}\right)\cdot n(n-1)$	$\frac{2}{15}+\frac{1519}{900\pi^2}+\frac{73763833}{3^4 5^4 2^6\pi^4}$	$\left(\frac{1}{45}-\frac{49}{54\pi^2}+\frac{1027741}{3^3 5^4 2^3\pi^4}\right)n^2+\left(\frac{1}{5}+\frac{441}{200\pi^2}-\frac{73763833}{3^4 5^4 2^6\pi^4}\right)n$	$n(n-1)\left[\left(\frac{1}{45}-\frac{49}{54\pi^2}+\frac{1027741}{3^3 5^4 2^3\pi^4}\right)n+\left(\frac{1}{15}+\frac{8869}{3600\pi^2}-\frac{29552237}{5^3 3^4 2^7\pi^4}\right)\right]$

independent of the event that chord 3 intersects chord 1. Each of these events has a probability of $2x(1-x)$. Thus

$$E(\alpha_{12}\alpha_{13}|x)=4x^2(1-x)^2$$

and

$$E(\alpha_{12}\alpha_{13})=\int_0^1 4x^2(1-x)^2\,dx=\frac{2}{15}$$

as before.

With this same approach we can obtain the probability mass function of w, the number of chords intersecting one particular chord, which we shall call C. If C intercepts an arc of length x $(0\leqq x\leqq 1)$, the probability of another chord intersecting C is, as before, $2x(1-x)$. The probability of another chord not intersecting C is $1-2x(1-x)$, which can be written as $x^2+(1-x)^2$. Thus the probability that among n random chords exactly w of them intersect C is

$$\begin{aligned}p(w|x)&=\binom{n}{w}2^w x^w(1-x)^w[x^2+(1-x)^2]^{n-w}\\&=\binom{n}{w}2^w x^w(1-x)^w\sum_{k=0}^{n-w}\binom{n-w}{k}x^{2k}(1-x)^{2(n-w-k)}.\end{aligned}$$

The probability that, regardless of the length x of the arc intercepted by C, exactly w chords intersect C is

$$\begin{aligned}p(w)&=\binom{n}{w}2^w\sum_{k=0}^{n-w}\binom{n-w}{k}\int_0^1 x^{2k+w}(1-x)^{2n-w-2k}\,dx\\&=\binom{n}{w}2^w\sum_{k=0}^{n-w}\binom{n-w}{k}\frac{(2k+w)!(2n-w-2k)!}{(2n+1)!}\\&=\frac{\binom{n}{w}2^w}{2n+1}\sum_{k=0}^{n-w}\binom{n-w}{k}\Big/\binom{2n}{2k+w}\qquad (w=0,1,2,\cdots,n).\end{aligned}$$

Another form of this expression given by David and Fix (1964) is

$$p(w)=\frac{1}{(n+1)2^{n-2w}}\sum_{k=0}^{n-w}\binom{2k}{k}\binom{n+1}{w+k+1}\Big/\binom{2w+2k+1}{w+k}\qquad (w=0,1,2,\cdots,n).$$

It is rather surprising that the distribution of w is totally unaffected by the shape of the convex region.

The probability mass function of r is not known at present except for a few special values of r when the random chords are defined as in Model I. The values of $P[r=0]$, $P[r=1]$, $P[r=(n(n-1)/2)-1]$ and $P[r=n(n-1)/2]$ for Model I were derived by Whitworth. The key to the derivation is illustrated in Fig. 6.16 below.

The two regions have different shapes and the endpoints of the chords are spaced differently on the perimeters of the regions. On the other hand the

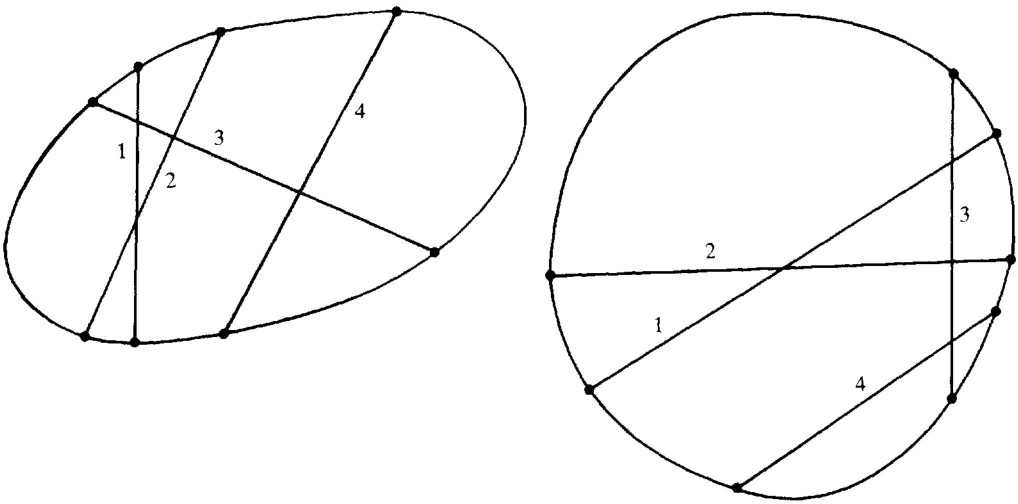


FIG. 6.16

configurations of the chords in the two regions are identical in a certain sense; namely, the order of the endpoints around the perimeter is 3, 1, 2, 4, 3, 4, 1, 2 for both regions. Clearly the number of intersections, r, is completely determined by the order of the endpoints around the perimeter and is unaffected by the shape of the convex region or the spacing of the endpoints on the perimeter. Thus without loss of generality we can fix $2n$ points on the perimeter of the region to be the endpoints of the chords. Under Model I each pair of points is equally likely to be connected by a chord. (This is not true for any of the other randomization models.) Thus we can compute $P[r=j]$ by finding the number of ways our $2n$ points can be connected forming chords with j intersections and dividing this number by the total number of ways the $2n$ points can be joined to form n chords.

First we shall derive the number of ways that $2n$ points on the circumference of a closed convex curve can be connected to form n chords. From the first given point we can draw a chord in $2n-1$ ways, from the next point in $2n-3$ ways (since the first chord used up two points), from the next in $2n-5$ ways, and so on. Hence the number of ways is

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdots 2n}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(2n)!}{n!2^n}.$$

We shall now derive $P[r=0]$. Let a_n denote the number of ways that $2n$ points on a closed convex curve can be connected to form n chords with no intersections within the region. Clearly $a_1 = 1$. Any chord must divide the remaining $2n-2$ points so that an even number remain on either side, say $2k$ on the left side and $2n-2k-2$ on the right side. The $2k$ points on the left can then

be joined to form k nonintersecting chords in a_k ways, and the $2n-2k-2$ points on the right can be joined to form $n-k-1$ nonintersecting chords in a_{n-k-1} ways. Thus if the first chord leaves $2k$ points on its left side, the remaining chords can be drawn in $a_k a_{n-k-1}$ ways. (If $k=0$ or $k=n-1$ then all of the remaining chords are on one side of the first chord and they can be drawn in a_{n-1} ways.) Since k can have any value from 0 to $n-1$ we have for $n>1$

$$a_n = a_{n-1} + a_1 a_{n-2} + a_2 a_{n-3} + \cdots + a_{n-2} a_1 + a_{n-1}.$$

The expression on the right hand side is the coefficient of x^{n-1} in the power series $(1+a_1x+a_2x^2+\cdots)^2$. Therefore

$$a_1 + a_2 x + a_3 x^2 + \cdots = (1 + a_1 x + a_2 x^2 + \cdots)^2.$$

Let S denote $(1+a_1x+a_2x^2+\cdots)$. Then the above equation can be written

$$\frac{S-1}{x} = S^2 \quad \text{or} \quad xS^2 - S + 1 = 0.$$

The solution is

$$S = \frac{1 \pm \sqrt{1-4x}}{2x}$$

We shall choose the minus sign as the plus sign yields a nonsensical answer. The series expansion for $\sqrt{1-4x}$ is

$$(1-4x)^{1/2} = 1 - \frac{1}{2}(4x) + \frac{\frac{1}{2}(\frac{1}{2}-1)(4x)^2}{2!} - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(4x)^3}{3!} + \cdots.$$

The coefficient of x^n in this expansion is (for $n \geqq 2$)

$$\begin{aligned}
c_n &= \frac{(-1)^n 4^n (\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n+1)}{n!} \\
&= \frac{(-1)^n 2^n (1-2)(1-4)(1-6)\cdots(1-2(n-1))}{n!} \\
&= \frac{(-1)^n(-1)^{n-1}2^n(1\cdot 3\cdot 5\cdots(2n-3))}{n!} \\
&= -\frac{2^n(2n-2)!}{n!\,2\cdot 4\cdot 6\cdots(2n-2)} \\
&= -\frac{2(2n-2)!}{n!(n-1)!}.
\end{aligned}$$

Hence

$$S=\frac{1-\sqrt{1-4x}}{2x}$$

$$=\frac{1-1+2x+\sum_{n=2}^{\infty}\frac{2(2n-2)!}{n!(n-1)!}x^n}{2x}$$

$$=1+\sum_{n=1}^{\infty}\frac{(2n)!}{(n+1)!n!}x^n.$$

Substituting in the definition of S we have

$$1+\sum_{n=1}^{\infty}a_nx^n=1+\sum_{n=1}^{\infty}\frac{(2n)!}{(n+1)!n!}x^n.$$

Equating the coefficients of x^n we get

$$a_n=\frac{(2n)!}{(n+1)!n!}.$$

We get $P[r=0]$ by dividing a_n, the number of ways of drawing n nonintersecting chords, by the total number of ways to draw n chords, i.e.,

$$P[r=0]=\frac{(2n)!}{(n+1)!n!}\bigg/\frac{(2n)!}{n!2^n}=\frac{2^n}{(n+1)!}.$$

In order that there is one and only one intersection among the n chords, we must have two chords which intersect and which are not intersected by any of the other $n-2$ chords none of which intersect each other. The two chords which intersect must divide the remaining $2n-4$ points into four groups each containing an even number. Chords must be drawn connecting the points in each of these groups so that there are no intersections. Thus if there are $2k$ points in a group, there are a_k ways of connecting the points so there are no intersections among the chords of that group. Thus if there are $2k$ points in the first group, $2j$ in the second, $2i$ in the third and $2m$ in the fourth (so that $2k+2j+2i+2m=2n-4$), then there are $a_ka_ja_ia_m$ ways of drawing the remaining $n-2$ chords so they have no intersections with each other or with the two intersecting chords. If we draw one of the intersecting chords through a particular point A from among the $2n$ points on the circumference, the number of ways of drawing the remaining chords is $\sum_{k+j+i+m=n-2}a_ka_ja_ia_m$. The point A can be chosen in $2n$ ways, but each arrangement will then be counted four times. (For example, the arrangement shown in Fig. 6.17 is counted once for $A=1$, $A=2$, $A=5$ and $A=6$). Thus the total number of ways of drawing the n chords with one intersection is $(n/2)\sum_{k+j+i+m=n-2}a_ka_ja_ia_m$, which is $n/2$ times the coefficient of x^{n-2} in $(1+a_1x+a_2x^2+a_3x^3+\cdots)^4$, i.e. S^4. In the derivation of $P[r=0]$ we found that

$$S^2=\frac{S-1}{x}.$$

Thus

$$S^4 = \frac{S^2 - 2S + 1}{x^2} = \frac{S - 1 - 2xS + x}{x^3}.$$

We also found that

$$S = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n.$$

Hence

$$\begin{aligned}
S^4 &= \frac{S - 1 - 2xS + x}{x^3} \\
&= \frac{1}{x^3}\left[1 + \left(x + \sum_{n=2}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n\right) - 1 - 2x - 2\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!} x^{n+1} + x\right] \\
&= \frac{1}{x^3}\left[\sum_{n=1}^{\infty}\left(\frac{(2n+2)!}{(n+1)!(n+2)!} - 2\frac{(2n)!}{n!(n+1)!}\right)x^{n+1}\right] \\
&= 2\sum_{n=2}^{\infty} \frac{(n-1)(2n)!}{n!(n+2)!} x^{n-2}.
\end{aligned}$$

Thus the coefficient of x^{n-2} in $(n/2)S^4$ is $n(n-1)(2n)!/(n!(n+2)!)$. This is the number of ways the n chords can be drawn with one intersection. Thus

$$P[r = 1] = \frac{n(n-1)(2n)!}{n!(n+2)!} \bigg/ \frac{(2n)!}{n!2^n} = \frac{2^n n(n-1)}{(n+2)!}.$$

The values of $P[r = j]$ for $1 < j < n(n-1)/2 - 1$ are not known. We can derive $P[r = n(n-1)/2 - 1]$ as follows. In order that r be $n(n-1)/2 - 1$ (i.e., one less

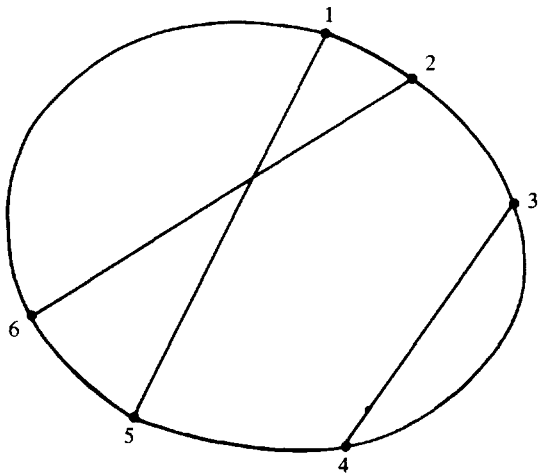


FIG. 6.17

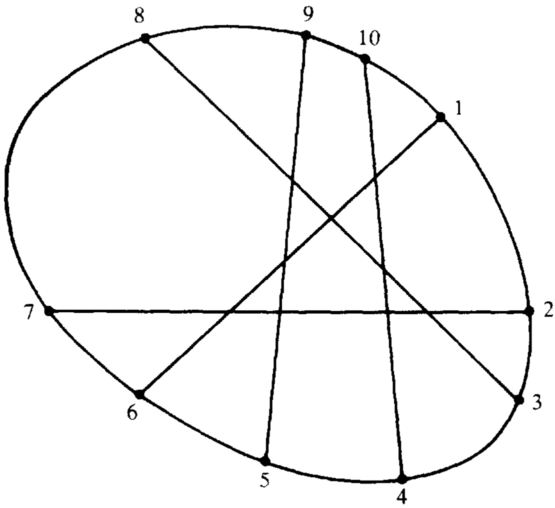


FIG. 6.18

than the maximum possible number of intersections), we must have two chords which do not intersect but the other $n-2$ chords must all intersect each other as well as the two nonintersecting chords. This can be done only if the two nonintersecting chords are adjacent (see Fig. 6.18) and there are $n-2$ points on each side of the nonintersecting chords. Each of these remaining points must be connected with the point opposite it on the boundary. (If the points are numbered consecutively then point 1 must be joined to point $n+1$, point 2 to point $n+2$ and so on, except for the endpoints of the nonintersecting chords.) Thus once the two nonintersecting chords have been chosen there is only one way to connect the remaining $2n-4$ points so that the $n-2$ chords all intersect the two nonintersecting chords and each other. There are n different ways to choose the two nonintersecting chords. Thus there are n ways of joining the $2n$ points so that the chords have $n(n-1)/2-1$ intersections within the region. Hence $P[r=n(n-1)/2-1]$ is n divided by the total number of ways the $2n$ points can be joined, that is

$$P\left[r=\frac{n(n-1)}{2}-1\right]=\frac{nn!2^n}{(2n)!}.$$

For each chord to intersect every other chord, each of the $2n$ points must be joined to the point opposite it. That is, if the points are numbered sequentially around the boundary, point 1 must be joined to point $n+1$, point 2 to point $n+2$, and so on. There is, of course, just one way (out of the $(2n)!/(n!2^n)$ possible ways of joining the $2n$ points) that this can be achieved. Thus the probability that the total number of intersections within the convex region is $n(n-1)/2$ is

$$P\left[r=\frac{n(n-1)}{2}\right]=\frac{n!2^n}{(2n)!}.$$

The above methods for deriving $P[r=0]$, $P[r=1]$, $P[r=(n(n-1)/2)-1]$ and $P[r=n(n-1)/2]$ are applicable only to Model I and do not extend to the other models. On the other hand, some of the above results, namely $E(\alpha_{12})$ for an arbitrary convex region and the mass function of w (for the circular case), can be obtained for Model II also. First we must restate the Model II randomization model so that it applies to any convex region C. Let O be any point in C. If the perpendicular from O to the extension of the random chord has length p and makes an angle ϕ with some axis, we want the joint density of p and ϕ to be proportional to $dp\, d\phi$ as in the circular case. (See Fig. 6.19.) In our previous discussion of random lines of a Poisson field which intersect a convex region, we found that the density of such lines is $L^{-1}\, dp\, d\phi$ where L is the perimeter of the region. Thus we see that random chords in a convex region C as defined by Model II must have the same distribution as random chords formed by random lines of a Poisson field intersecting C. We have already obtained several results for random chords formed by random lines intersecting a convex region. The probability that two chords intersect within C is

$$E(\alpha_{12})=\frac{2\pi A}{L^2}$$

where A is the area and L the perimeter of C. For the case where C is a circle we get $E(\alpha_{12})=\frac{1}{2}$, which agrees with our previous result.

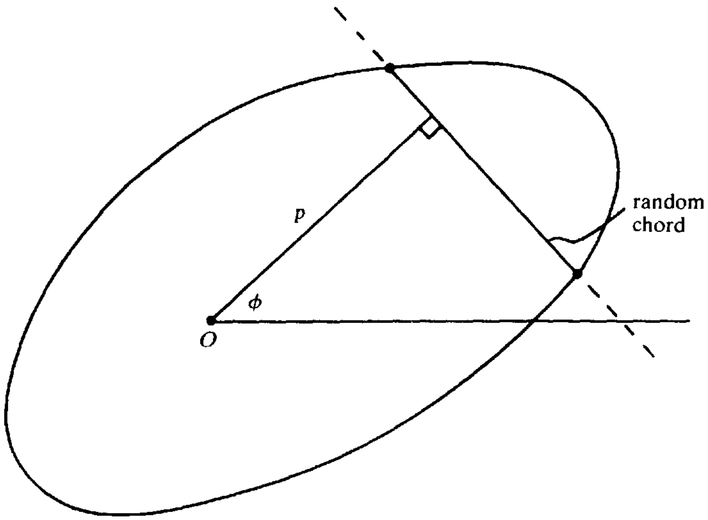


FIG. 6.19

Another result from the section on random lines intersecting a convex region is the fact that the probability that a random chord intersects a line segment of length l in C is $2l/L$. Thus if a random chord in a circle of unit radius has length l, the probability that it is hit by another random chord is l/π. We shall use this fact to derive the mass function of w for Model II in the circular case. The

method is quite similar to that used above in the derivation of the mass function of w for Model I. If p is the distance from the random chord to the center of the circle, p is distributed uniformly on [0, 1] and $l = 2\sqrt{1-p^2}$. Thus we have

$$P[\text{Intersection} \mid p] = \frac{2\sqrt{1-p^2}}{\pi}$$

and

$$P[\text{No Intersection} \mid p] = 1 - \frac{2}{\pi}\sqrt{1-p^2}.$$

The probability that n random chords intersect a chord of length l exactly w times is

$$\begin{aligned} p(w \mid p) &= \binom{n}{w}\left(\frac{2}{\pi}\right)^w (1-p^2)^{w/2}\left[1 - \frac{2}{\pi}\sqrt{1-p^2}\right]^{n-w} \\ &= \binom{n}{w}\left(\frac{2}{\pi}\right)^w (1-p^2)^{w/2} \sum_{k=0}^{n-w} \binom{n-w}{k}\left(-\frac{2}{\pi}\right)^{n-w-k} (1-p^2)^{(n-w-k)/2} \\ &= \binom{n}{w} \sum_{k=0}^{n-w} (-1)^{n-w-k}\binom{n-w}{k}\left(\frac{2}{\pi}\right)^{n-k} (1-p^2)^{(n-k)/2}. \end{aligned}$$

Thus the unconditional probability mass function of w is

$$p(w) = \binom{n}{w} \sum_{k=0}^{n-w} (-1)^{n-w-k}\binom{n-w}{k}\left(\frac{2}{\pi}\right)^{n-k} \int_0^1 (1-p^2)^{(n-k)/2}\, dp.$$

Letting $x = p^2$, we get $dp = \frac{1}{2}x^{-1/2}\, dx$ and

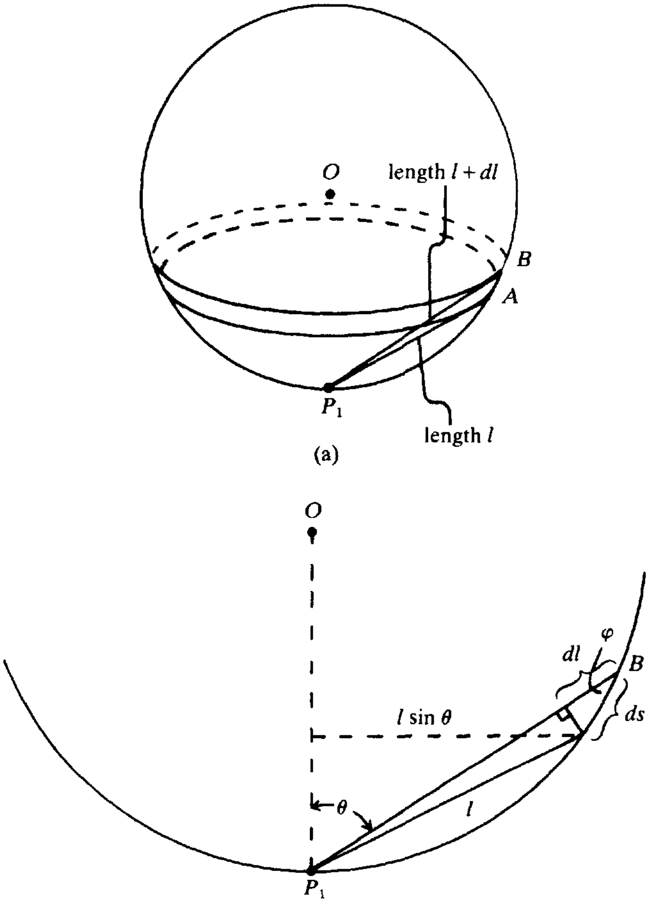
$$\begin{aligned} p(w) &= \frac{1}{2}\binom{n}{w} \sum_{k=0}^{n-w} (-1)^{n-w-k}\binom{n-w}{k}\left(\frac{2}{\pi}\right)^{n-k} \int_0^1 x^{-1/2}(1-x)^{(n-k)/2}\, dx \\ &= \frac{1}{2}\binom{n}{w} \sum_{k=0}^{n-w} (-1)^{n-w-k}\binom{n-w}{k}\left(\frac{2}{\pi}\right)^{n-k} \frac{\Gamma(1/2)\Gamma((n-k+2)/2)}{\Gamma((n-k+3)/2)}. \end{aligned}$$

Hence we have shown that the probability that a chord chosen from among $n+1$ random (Model II) chords of a circle is crossed exactly w times by the remaining n chords is

$$p(w) = \frac{\sqrt{\pi}}{2}\binom{n}{w} \sum_{k=0}^{n-w} (-1)^{n-w-k}\binom{n-w}{k}\left(\frac{2}{\pi}\right)^{n-k} \frac{\Gamma((n-k+2)/2)}{\Gamma((n-k+3)/2)}.$$

Random chords of a sphere. In this section, we will consider the distribution of the length of a random chord of a sphere, under eight different randomization models. The developments are due to Berengut (1972). This treatment will be somewhat analogous to the discussion of random chords of a circle, under various models, considered earlier in connection with the paper of David and Fix (1964). We will assume in all cases that the sphere has unit radius, and we will let L be the random variable denoting chord length, with density $f(l)$.

Model 1. Chord joining two points uniformly distributed on surface of sphere. We can let the position P_1 of the first point be arbitrary. Then the $\Pr\{l<L<l+dl\}$ is $1/(4\pi)$ times the area of the circular band in Fig. 6.20.



(b) *Cross-section through* P_1.

FIG. 6.20

Now the width of the band is $ds = \sec\varphi\, dl$. But φ is the angle between P_1B and the tangent to the circle at B. Hence by elementary geometry, φ is equal to the angle subtended on the upper part of the circle by the chord P_1B. But this angle is just $(\pi/2)-\theta$. Hence $\varphi=(\pi/2)-\theta$, and therefore $ds = \csc\theta\, dl$. Now the area of the band is ds times its circumference, which is $2\pi \cdot l\sin\theta$. Therefore,

$$\text{(6.56a)} \qquad \text{Area of band} = \cos\theta\, dl \cdot 2\pi l \sin\theta = 2\pi l\, dl;$$

$$\text{(6.56b)} \qquad \Pr\{l<L<l+dl\} = \frac{1}{4\pi}\cdot 2\pi l\, dl = \frac{l}{2}\, dl;$$

and

(6.56c) $$f(l)=l/2, \qquad 0\leqq l\leqq 2,$$

(6.56d) $$E(l)=\int_0^2 \frac{l^2}{2}\,dl=\left[\frac{l^3}{6}\right]_0^2=\frac{4}{3},$$

(6.56e) $$\operatorname{Var}(l)=\int_0^2 \frac{l^3}{2}\,dl-\frac{16}{9}=\left[\frac{l^4}{8}\right]_0^2-\frac{16}{9}=\frac{2}{9}.$$

Model 2. Chord in random direction from point uniformly distributed on surface. Let the position of the point be P. Let θ be the angle between the random chord through P and the diameter through P. (See Fig. 6.21.) Since the chord PQ is randomly directed, we know from a previous result that the density of θ is proportional to $\sin\theta$, i.e., θ has density $g(\theta)=\sin\theta$, $0\leqq\theta\leqq\pi/2$. Now triangle PQR is a right triangle with hypotenuse of length 2; hence $l=2\cos\theta$. Thus $|dl/d\theta|=2\sin\theta$. The density of l is thus $f(l)=g(\theta)/|dl/d\theta|=\sin\theta/(2\sin\theta)=1/2$; i.e.,

(6.57) $$f(l)=\frac{1}{2}, \qquad 0\leqq l\leqq 2,$$
$$E(l)=1, \qquad \operatorname{Var}(l)=\frac{1}{3}.$$

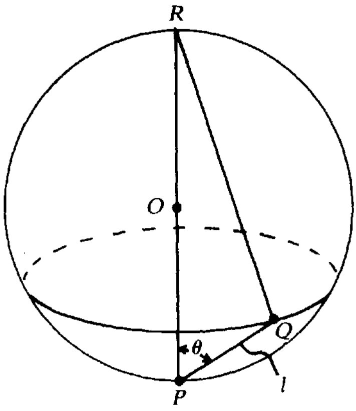


FIG. 6.21

Model 3. Chord in random direction, distance from center uniformly distributed. Let r be the distance of the chord from the center of the circle. Then

$$l=2\sqrt{1-r^2}; \qquad \left|\frac{dl}{dr}\right|=\left|\frac{2(-2r)}{2\sqrt{1-r^2}}\right|=\frac{2r}{\sqrt{1-r^2}}.$$

Since r is uniformly distributed on (0, 1),

$$f(l)=\frac{1}{2r/\sqrt{1-r^2}}=\frac{\sqrt{1-r^2}}{2r}=\frac{l/2}{2\sqrt{1-l^2/4}}=\frac{l}{2\sqrt{4-l^2}},\qquad 0\leqq l\leqq 2\cdots,$$

(6.58)

$$E(l)=\frac{1}{2}\int_0^2\frac{l^2}{\sqrt{4-l^2}}\,dl.$$

Let $u=(4-l^2)/4$. Then $l=2\sqrt{1-u}$, $du=-(l/2)\,dl$; therefore,

$$\begin{aligned}E(l)&=\frac{1}{2}\int_0^2\frac{4(1-u)}{2\sqrt{u}}\frac{-2}{2\sqrt{1-u}}\,du\\&=\int_0^1 u^{-1/2}(1-u)^{1/2}\,du\\&=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}=\frac{1}{2}\sqrt{\pi}\sqrt{\pi}=\frac{\pi}{2},\end{aligned}$$

$$E(l^2)=2\int_0^1 u^{-1/2}(1-u)\,du=\frac{2\Gamma(\frac{1}{2})\Gamma(2)}{\Gamma(\frac{5}{2})}=\frac{2\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}=\frac{8}{3};$$

therefore,

$$\operatorname{Var}(l)=\frac{8}{3}-\frac{\pi^2}{4}.$$

Model 4. Center of chord uniformly distributed inside sphere; random direction. Let r be the distance of the chord (i.e., the center of the chord) from the center of the sphere.

$$\begin{aligned}\Pr(r_0\leqq r\leqq r_0+dr)&=\Pr(r\leqq r_0+dr)-\Pr(r\leqq r_0)\\&=\frac{1}{\frac{4}{3}\pi}\left(\frac{4}{3}\pi(r_0+dr)^3\right)-\frac{1}{\frac{4}{3}\pi}\left(\frac{4}{3}\pi r_0^3\right)\\&=(r_0+dr)^3-r_0^3\\&=3r_0^2\,dr+o(dr).\end{aligned}$$

Hence r has density

(6.59) $$g(r)=3r^2,\qquad 0\leqq r\leqq 1.$$

Now $l=2\sqrt{1-r^2}$, $r=\sqrt{(4-l^2)/2}$, $|dl/dr|=2r/\sqrt{1-r^2}$. Hence the density of l is given by

$$f(l)=3r^2\frac{\sqrt{1-r^2}}{2r}=\frac{3}{2}\pi\sqrt{1-r^2}=\frac{3}{2}\frac{\sqrt{4-l^2}}{2}\cdot\frac{l}{2}=\frac{3}{8}l\sqrt{4-l^2},$$

(6.60)

$$E(l)=\frac{3}{8}\int_0^2 l^2\sqrt{4-l^2}\,dl,\qquad 0\leqq l\leqq 2.$$

As before, letting $u=(4-l^2)/4$, we get

$$E(l)=\frac{3}{8}\int_0^1 4(1-u)\cdot 2u^{1/2}\cdot\frac{2}{2(1-u)^{1/2}}\,du=3\int_0^1 u^{1/2}(1-u)^{1/2}\,du$$

$$=\frac{3\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)}=\frac{3\cdot(\pi/4)}{2}=\frac{3\pi}{8}.$$

Similarly,

$$E(l^2)=6\int_0^1 u^{1/2}(1-u)\,du=\frac{6\Gamma(\frac{3}{2})\Gamma(2)}{\Gamma(\frac{7}{2})}=\frac{6\cdot\frac{1}{2}\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}=\frac{8}{5},$$

$$\operatorname{Var}(l)=\frac{8}{5}-\frac{9\pi^2}{64}.$$

Model 5. Chord in random direction through point uniformly distributed inside sphere. Let t be the distance of the random point P from the center of the sphere. Let θ be the acute angle between the chord and the radius through P. Then the distance from the center to the chord is $r=t\sin\theta$. Figure 6.22, which gives the cross-section through the sphere containing the chord and the center of the sphere, illustrates this. Now, since the chord through P has random direction, we know that θ has density $g(\theta)=\sin\theta$, $0\leqq\theta\leqq\pi/2$. Let t be given. Then $dr=t\cos\theta\,d\theta$. Thus the conditional density of r, given t is

$$h(r\mid t)=\sin\theta\frac{1}{t\cos\theta}=\frac{\tan\theta}{t}=\frac{r}{t\sqrt{t^2-r^2}},\qquad 0\leqq r\leqq t. \tag{6.61}$$

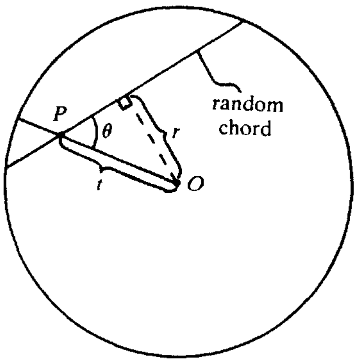


FIG. 6.22

Now since P is uniformly distributed inside the sphere, t has density $g(t)=3t^2$, $0\leqq t\leqq 1$, as was demonstrated previously in Model 4. Hence the density of r is

$$h(r)=\int_0^1 h(r\mid t)g(t)\,dt=\int_r^1\frac{r}{t\sqrt{t^2-r^2}}3t^2\,dt=3r\int_r^1\frac{t}{\sqrt{t^2-r^2}}\,dt$$

$$=\frac{3r}{2}\int_0^{1-r^2}\frac{du}{\sqrt{u}}=\frac{3r}{2}[2u^{1/2}]_0^{1-r^2}=3r\sqrt{1-r^2},$$

that is,

$$h(r)=3r\sqrt{1-r^2}, \qquad 0\leqq r\leqq 1. \tag{6.62}$$

Now $l=2\sqrt{1-r^2}$, $|dl/dr|=2r/\sqrt{1-r^2}$. Therefore,

$$f(l)=3r\sqrt{1-r^2}\cdot\frac{\sqrt{1-r^2}}{2r}=\frac{3}{2}(1-r^2)=\frac{3}{8}l^2, \qquad 0\leqq l\leqq 2,$$

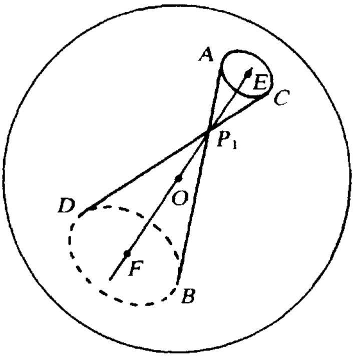
$$E(l)=\frac{3}{8}\int_0^2 l^3\,dl=\frac{3}{32}\cdot 16=\frac{3}{2}, \tag{6.63a}$$

$$E(l^2)=\frac{3}{8}\int_0^2 l^4\,dl=\frac{3}{40}\cdot 32=\frac{12}{5};$$

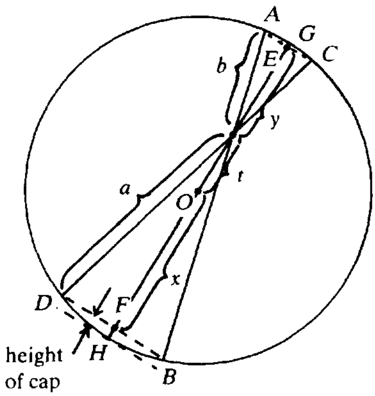
therefore

$$\operatorname{Var}(l)=\frac{12}{5}-\frac{9}{4}=\frac{3}{20}. \tag{6.63b}$$

Model 6. Chord through two points independently uniformly distributed inside sphere. Let the position of one of the points P_1 be at a distance t from the center. We will first condition on t. Consider $\Pr(L\geqq l\,|\,t)$, for $2\sqrt{1-t^2}\leqq l\leqq 2$. This is just $1/(4\pi)$ times the volume inside a pair of cones inside the sphere, with axis the diameter through P_1. Figure 6.23 illustrates the region.



(a)



(b) *Cross-section containing A, B, C, D.*

FIG. 6.23

We have $|P_1C|=|P_1A|=b$, $|P_1B|=|P_1D|=a$, $a+b=l$. Also, $|P_1E|=y$, $|OF|=x$. (If F lies on the same side of O as P_1, then we take x to be negative.) It is clear that $L\geqq l$ if and only if the second point P_2 lies inside the regions bounded by the sides of the cones. Now the region above P_1 (having A and C as boundary points) consists of a cone with flat base of height y, plus a spherical cap

of height $1-t-y$. Similarly, the region below P_1 (having D and B as boundary points) consists of a cone of height $x+t$ and a spherical cap of height $1-x$. We want to obtain the volume of these regions in terms of t and l.

Consider a spherical cap of height h, $0 \leqq h \leqq 1$. Its base is a circle of radius $\sqrt{1-(1-h)^2}$. (See Fig. 6.24.)

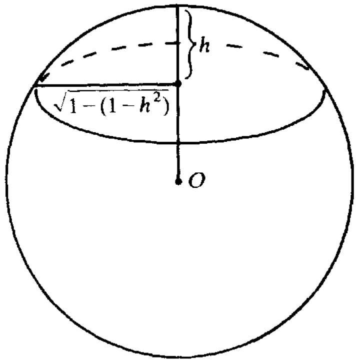


FIG. 6.24

Now, clearly, the volume of the cap is given by

$$V(h) = \int_0^{2\pi} \int_0^{\sqrt{1-(1-h)^2}} [\sqrt{1-r^2} - (1-h)] r \, dr \, d\theta$$

$$= 2\pi \frac{1}{2} \left[\frac{(1-r^2)^{3/2}}{3/2} \right]_{r=\sqrt{1-(1-h)^2}}^{r=0} - 2\pi(1-h) \frac{1-(1-h)^2}{2}$$

$$= \pi \left[\frac{2}{3}(1-(1-h)^3) - (1-h) + (1-h)^3 \right]$$

$$= \pi \left[\frac{2}{3} - (1-h) + \frac{1}{3}(1-h)^3 \right], \qquad 0 \leqq h \leqq 1.$$

Now for $1 \leqq h \leqq 2$, $V(h) = (4/3)\pi - V(2-h) = \pi((2/3) - (1-h) + (1/3)(1-h)^3)$. Thus

$$V(h) = \pi\left(\frac{2}{3} - (1-h) + \frac{1}{3}(1-h)^3\right), \qquad 0 \leqq h \leqq 2. \tag{6.64}$$

Hence the volume of the upper cap is

$$V_1 = \pi\left(\frac{2}{3} - (y+t) + \frac{(y+t)^3}{3}\right). \tag{6.65}$$

Now, for the lower cap, since we take x to be negative for $h > 1$, we have $1-h = x$ in all cases. Hence the volume of the lower cap is

$$V_2 = \pi\left(\frac{2}{3} - x + \frac{x^3}{3}\right). \tag{6.66}$$

Now the upper cone has base radius $\sqrt{b^2-y^2}$ and height y; hence its volume is

$$V_3=\frac{1}{3}\pi(b^2-y^2)y. \tag{6.67}$$

Similarly, the lower cone has volume

$$V_4=\frac{1}{3}\pi(1-x^2)(t+x). \tag{6.68}$$

Adding (6.65)–(6.68), we get

$$V=V_1+V_2+V_3+V_4=\pi\left[\frac{4}{3}+\left(\frac{b^2}{3}+t^2-1\right)y+ty^2-\frac{2}{3}x-\frac{t}{3}x^2-\frac{2t}{3}+\frac{t^3}{3}\right]. \tag{6.69}$$

We want to express V in terms of t and l. To do this, we have to find expressions for a, b, x, and y in terms of t and l.

Now, from elementary geometry, we know that for two intersecting chords of a circle, the product of the segments of one is equal to the product of the segments of the other. Applying this to chords AB and GH, we get $ab=(1-t)(1+t)$. Now using the relation $a+b=l$, we can solve a quadratic equation to obtain the relations.

$$a=\frac{l+\sqrt{l^2-4(1-t^2)}}{2}, \tag{6.70}$$

$$b=\frac{l-\sqrt{l^2-4(1-t^2)}}{2}. \tag{6.71}$$

We also have the relations $b^2-y^2=1-(y+t)^2$ and $1-x^2=a^2-(t+x)^2$, from which we obtain the following expression for x and y:

$$x=\frac{a^2-t^2-1}{2t}, \tag{6.72}$$

$$y=\frac{1-t^2-b^2}{2t}. \tag{6.73}$$

Using (6.70) and (6.71), we obtain finally

$$x=\frac{l^2+4+l\sqrt{l^2-4(1-t^2)}}{4t}, \tag{6.74}$$

$$y=\frac{4-l^2+l\sqrt{l^2-4(1-t^2)}}{4t}-t. \tag{6.75}$$

Substituting these expressions into (6.69), after considerable simplification we obtain

$$V=\pi\left(\frac{4}{3}-\frac{l^3\sqrt{l^2-4(1-t^2)}}{12t}\right), \qquad 2\sqrt{1-t^2}\leqq l\leqq 2. \tag{6.76}$$

Now since $\Pr(L>l\,|\,t)=(4\pi/3)^{-1}V$, for $2\sqrt{1-t^2}\leqq l\leqq 2$, we have:

$$\Pr(L>l\,|\,t)=\begin{cases}1-\dfrac{l^3\sqrt{l^2-4(1-t^2)}}{16t}, & 2\sqrt{1-t^2}\leqq l\leqq 2,\\ 1, & 0\leqq l\leqq 2\sqrt{1-t^2}.\end{cases}\tag{6.77}$$

Now t, being the distance of P_1 from the center O, has density $f(t)=3t^2$, as we saw before in (6.59). Hence

$$\begin{aligned}\Pr(L>l)&=\int_0^1 \Pr(L>l\,|\,t)3t^2\,dt\\ &=\int_0^1 3t^2\,dt-\int_{\sqrt{(4-l^2)}/2}^1 \frac{l^3\sqrt{l^2-4(1-t^2)}}{16t}3t^2\,dt\\ &=1-\frac{3l^3}{16}\cdot\int_{\sqrt{(4-l^2)}/2}^1 t\sqrt{l^2-4(1-t^2)}\,dt\\ &=1-\frac{3l^3}{16}\cdot\frac{1}{8}\cdot\frac{2}{3}[\{l^2-4(1-t^2)\}^{3/2}]_{t=\sqrt{(4-l^2)}/2}^{t=1}\\ &=1-\frac{l^6}{64}\end{aligned}\tag{6.78}$$

Hence $\Pr(L\leqq l)=l^6/64$, $0\leqq l\leqq 2$, and the density of L is

$$f(l)=\frac{3}{32}l^5,\qquad 0\leqq l\leqq 2,\tag{6.79a}$$

$$E(l)=\frac{3}{32}\int_0^2 l^6\,dl=\frac{3}{32}\left(\frac{2^7}{2}\right)=\frac{12}{7},\tag{6.79b}$$

$$E(l^2)=\frac{3}{32}\int_0^2 l^7\,dl=\frac{3}{32}\left(\frac{2^8}{8}\right)=3;\tag{6.79c}$$

therefore

$$\operatorname{Var}(l)=3-\frac{144}{49}=\frac{3}{49}.\tag{6.79d}$$

Model 7. Chord normal to random plane intersecting sphere, through point uniformly distributed inside circle of intersection. By random planes we mean planes with measure invariant under rotations, translations and reflections. We are interested in the conditional space of random planes which intersect the sphere. Let p be the distance of the random intersecting plane from the center of the sphere. The random chord is obtained by choosing a point P having uniform distribution in the circle of intersection of the plane with the sphere, and drawing the normal to the plane through that point. Let r be the distance of the point P from the center Q of the circle of intersection, $0\leqq r\leqq\sqrt{1-p^2}$. (See Fig. 6.25.)

Let p be given. Then

$$\Pr(r_0 \leqq r \leqq r_0 + dr) = \frac{1}{\pi(1-p^2)}[\pi(r_0+dr)^2 - \pi r_0^2]$$
$$= \frac{2r_0}{1-p^2}\,dr + o(dr), \qquad 0 \leqq r_0 \leqq \sqrt{1-p^2}.$$

Hence the conditional density of r given p is

$$g(r\,|\,p) = \frac{2r}{1-p^2}, \qquad 0 \leqq r \leqq 1-p^2. \tag{6.80}$$

Now since OQ and the chord are both normal to the same plane, the distance from the chord to the center O is r. Hence the length of the chord is $l = 2\sqrt{1-r^2}$, $|dl/dr| = 2r/\sqrt{1-r^2}$. Making the transformation to l, we obtain the conditional density of l:

$$f(l\,|\,p) = \frac{2r}{1-p^2}\cdot\frac{\sqrt{1-r^2}}{2r} = \frac{\sqrt{1-r^2}}{1-p^2} = \frac{l}{2(1-p^2)}, \qquad 2p \leqq l \leqq 2. \tag{6.81}$$

Now from the theory of random planes, we know (cf. Kendall and Moran (1963, pp. 20–22)) that p has the uniform distribution. Hence the density of l is

$$f(l) = \int_0^{l/2} f(l\,|\,p)\,dp = \int_0^{l/2} \frac{l}{2(1-p^2)}\,dp = \frac{l}{2}\int_0^{l/2}\left(\frac{1}{2(1+p)} + \frac{1}{2(1-p)}\right)dp$$
$$= \frac{l}{4}[\ln(1+p) - \log(1-p)]_{p=0}^{p=l/2}$$
$$= \frac{l}{4}\ln\frac{2+l}{2-l}, \qquad 0 \leqq l \leqq 2,$$

$$E(l) = \frac{1}{4}\int_0^2 l^2 \ln\frac{2+l}{2-l}\,dl = \frac{1}{4}\left\{\int_0^2 l^2\ln(2+l)\,dl - \int_0^2 l^2 \ln(2-l)\,dl\right\} \tag{6.82}$$
$$= \frac{1}{4}\left\{\int_{\ln 2}^{\ln 4} u(e^{3u} - 4e^{2u} + 4e^u)\,du - \int_{-\infty}^{\ln 2} u(e^{3u} - 4e^{2u} + 4e^u)\,du\right\}$$
$$= \frac{1}{4}\left\{\left[\frac{e^{3u}}{3}\left(u - \frac{1}{3}\right) - 4\frac{e^{2u}}{2}\left(u - \frac{1}{2}\right) + 4e^u(u-1)\right]_{u=\ln 2}^{u=\ln 4}\right.$$
$$\left. - \left[\frac{e^{3u}}{3}\left(u - \frac{1}{3}\right) - 4\frac{e^{2u}}{2}\left(u - \frac{1}{2}\right) + 4e^u(u-1)\right]_{\ln 2}^{\ln 4}\right\} = \frac{4}{3}\ln 2 + \frac{2}{3}.$$

In a similar manner, we obtain $E(l^2) = 8/3$. Hence, $\operatorname{Var}(l) = 20/9 - (16/9)\ln 2 - (16/9)(\ln 2)^2$.

Model 8. Chord normal to plane of random great circle, through point uniformly distributed in circle of intersection. This randomization model corresponds to the measure obtained by taking the restriction of "random" lines in

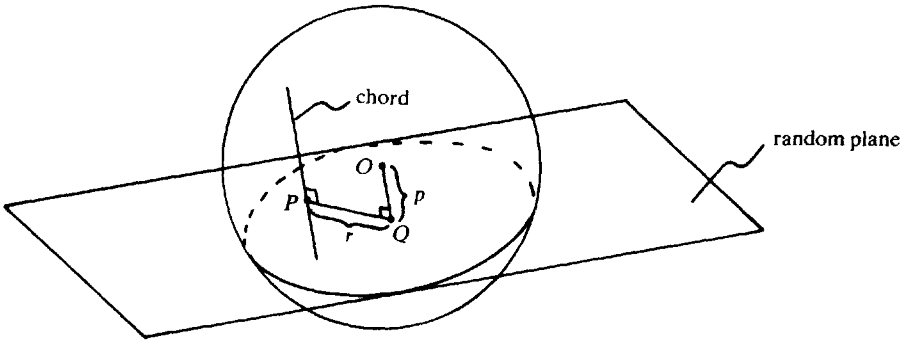


FIG. 6.25

3 dimensions to those intersecting the sphere, where "random" lines in 3 dimensions refers to the measure on lines which is invariant under translations, rotations and reflections.

Let r be the distance of the chord from the center. By construction, P is uniformly distributed inside the great circle. (See Fig. 6.26.) Hence, applying (6.80) with $p=0$, we find that the density of r is

(6.83) $$g(r)=2r, \qquad 0 \leqq r \leqq 1.$$

Making the transformation $l=2\sqrt{1-r^2}$, we obtain the density of l:

(6.84) $$f(l)=\frac{l}{2}, \qquad 0 \leqq l \leqq 2.$$

We observe that this is the same density as that obtained under Model 1, where the chord is formed by joining two points independently uniformly distributed on the surface of the sphere. Since the measures induced by both models are directionally homogeneous by construction, it is clear that the measures are identical.

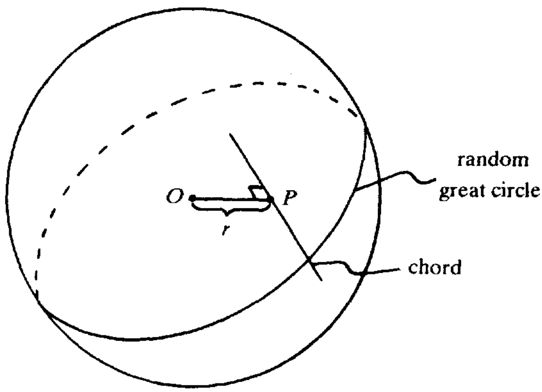


FIG. 6.26

We also observe that $E(l)$, which was shown to be $4/3$ in Model 1, indeed is equal to $4V/S$. Table 5 summarizes our results.

TABLE 5

Model	$f(l)$ $0 \leq l \leq 2$	$E(l)$	Var (l)
1. Joining two random points on surface	$\frac{l}{2}$	$\frac{4}{3} = 1.333$	$\frac{2}{9} = .333$
2. Random direction from random point on surface	$\frac{1}{2}$	$1 = 1.000$	$\frac{1}{3} = .333$
3. Random direction, uniformly distributed distance from center	$\frac{l}{2\sqrt{4-l^2}}$	$\frac{\pi}{2} = 1.571$	$\frac{8}{3} - \frac{\pi^2}{4} = .199$
4. Chord center uniformly distributed inside sphere; random direction	$\frac{3}{8} l \sqrt{4-l^2}$	$\frac{3\pi}{8} = 1.178$	$\frac{8}{5} - \frac{9\pi^2}{64} = .212$
5. Random direction through random point inside sphere	$\frac{3}{8} l^2$	$\frac{3}{2} = 1.500$	$\frac{3}{20} = .150$
6. Through 2 random points inside sphere	$\frac{3}{32} l^5$	$\frac{12}{7} = 1.714$	$\frac{3}{49} = .061$
7. Normal to random intersecting plane, through random point in circle of intersection	$\frac{l}{4} \ln \frac{2+l}{2-l}$	$\frac{4}{3} \ln 2 + \frac{2}{3} = 1.591$	$\frac{20}{9} - \frac{16}{9}(\ln 2 + \ln^2 2) = .134$
8. Normal to plane of random great circle, through random point in circle of intersection	$\frac{l}{2}$	same as	Model 1

References

H. A. ALIKOSKI (1939), *Über das Sylvestersche Vierpunktproblem*, Ann. Acad. Sci. Fenn., 51, no. 7, pp. 1–10.

D. E. BARTON, F. N. DAVID AND E. FIX (1963), *Random points in a circle and the analysis of chromosome patterns*, Biometrika, 50, pp. 23–30.

M. BATICLE (1935), *Le problème des répartitions*, Comptes Rendus, 201, pp. 862–864.

D. BERENGUT (1972), *Random chords of a sphere*, Tech. Rep. 191, Stanford Univ., Stanford, CA.

P. BILLARD (1965), *Sur les arcs places au hasard sur le cercle*, Ann. Sci. École Norm. Sup., 82, pp. 152–179.

G. BUFFON (1777), *Essai d'arithmétique morale*, Supplément a l'Histoire Naturelle, vol. 4.

H. CHALKLEY, J. CORNFIELD AND H. PARK (1949), *A method for estimating volume-surface ratios*, Science, 110, pp. 295–297.

P. J. COOKE (1972), *Sequential coverage in geometrical probability*, Tech. Rep. 198, Stanford Univ., Stanford, CA.

J. CORNFIELD AND H. W. CHALKLEY (1951), *A problem in geometric probability*, J. Washington Acad. Sci., 41, pp. 226–229.

M. W. CROFTON (1885), *Probability*, Encyclopaedia Britannica, 9th ed., vol. 19, pp. 768–788.

D. A. DARLING (1953), *On a class of problems related to the random division of an interval*, Ann. Math. Statist., 24, pp. 239–253.

F. N. DAVID AND EVELYN FIX (1964), *Intersections of random chords of a circle*, Biometrika, 51, pp. 373–379.

P. DIACONIS (1976), *Buffon's problem with a long needle*, J. Appl. Probability, 13, pp. 614–618.

C. DOMB (1947), *The problem of random intervals on a line*, Proc. Cambridge Philos. Soc., 43, pp. 329–341.

S. W. DUFOUR (1972), *Intersections of random convex regions*, Ph.D. dissertation, Dept. of Statistics, Stanford Univ., Stanford, CA.

A. DVORETSKY (1956), *On covering a circle by randomly placed arcs*, Proc. Nat. Acad. Sci. U.S.A., 42, pp. 199–203.

W. FELLER (1971), *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd ed., John Wiley, New York.

R. A. FISHER (1929), *Tests of significance in harmonic analysis*, Proc. Roy. Soc. London Ser. A, 125, pp. 54–59.

——— (1940), *On the similarity of the distributions found for the test of significance in harmonic analysis, and in Stevens's problem in geometrical probability*, Eugenics, 10, pp. 14–17.

H. FLANDERS (1963), *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York.

L. FLATTO AND A. G. KONHEIM (1962), *The random division of an interval and the random covering of a circle*, SIAM Rev., 4, pp. 211–222.

F. GARWOOD (1940), *An application to the theory of probability of the operation of vehicular-controlled traffic signals*, J. Roy. Statist. Soc., Suppl., 7, pp. 65–77.

E. N. GILBERT (1965), *The probability of covering a sphere with N circular caps*, Biometrika, 52, pp. 323–330.

S. GOUDSMIDT (1945), *Random distributions of lines in a plane*, Rev. Modern Phys., 17, pp. 321–322.

B. HOSTINSKY (1925), *Sur les probabilités géometriques*, Publ. Fac. Sci. Univ. Masaryk, Brno 50, Czechoslovakia.

M. G. KENDALL AND P. A. P. MORAN (1963), *Geometrical Probability*, Griffin, London.

J. F. C. KINGMAN (1969), *Random secants of a convex body*, J. Appl. Probability, 6, pp. 660–672.

M. MANDELBROT (1972), *On Dvoretsky coverings for the circle*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 22, pp. 158–160.

L. MANTEL (1953), *An extension of the Buffon needle problem*, Ann. Math. Statist., 24, pp. 674–677.

J. G. MAULDON (1951), *Random division of an interval*, Proc. Cambridge Philos. Soc., 47, pp. 331–336.

R. E. MILES (1964), *Random polygons determined by random lines in a plane, I, II*, Proc. Nat. Acad. Sci. USA, 52, pp. 901–907, 1157–1160.

P. A. P. MORAN AND S. FAZEKAS DE ST. GROTH (1962), *Random circles on a sphere*, Biometrika, 49, pp. 389–396.

R. R. A. MORTON (1966), *The expected number and angle of intersections between random curves in a plane*, J. Appl. Probability, 3, pp. 559–562.

M. PERLMAN AND M. WICHURA (1975), *On sharpening Buffon's needle*, Amer. Statist., 29, pp. 157–163.

S. RÉNYI (1964), *On two mathematical models of the traffic on a divided highway*, J. Appl. Probability, 1, pp. 311–320.

H. E. ROBBINS (1944), *On the measure of a random set*, Ann. Math. Statist., 14, pp. 70–74.

——— (1945), *On the measure of a random set. II*, Ibid., 16, pp. 342–347.

H. RUBEN AND W. J. REED (1973), *A more general form of the theorem of Crofton*, J. Appl. Probability, 10, pp. 479–482.

L. A. SANTALÓ (1953), *Introduction to Integral Geometry*, Actualités Scientifiques et Industrielles, No. 1198, Hermann, Paris.

——— (1976), *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications I, Addison-Wesley, Reading, MA.

E. F. SCHUSTER (1974), *Buffon's needle experiment*, Amer. Math. Monthly, 81, pp. 26–29.

L. A. SHEPP (1972), *Covering the circle with random arcs*, Israel J. Math., 11, pp. 328–345.

ANDREW F. SIEGEL (1977), *Random coverage problems in geometrical probability with an application to time series analysis*, Ph.D. dissertation, Stanford Univ., Stanford, CA.

H. SOLOMON AND PETER C. C. WANG (1972), *Nonhomogeneous Poisson fields of random lines with applications to traffic flow*, Proc. Sixth Berkeley Symposium on Math. Stat. and Prob. III, pp. 383–400.

W. L. STEVENS (1939), *Solution to a geometrical problem in probability*, Ann. Eugenics, 9, pp. 315–320.

J. V. USPENSKY (1937), *Introduction to Mathematical Probability*, McGraw-Hill, New York.

G. WEISS AND R. HERMAN (1962), *Statistical properties of low-density traffic*, Quart. Appl. Math., 20, pp. 121–130.

J. G. WENDEL (1962), *A problem in geometric probability*, Math. Scand., 2, pp. 109–111.

W. A. WHITWORTH (1897), *DCC Exercises in Choice and Chance* (Reprint of 1897 edition), Hafner, New York, 1965.

J. WOLFOWITZ (1949), *The distribution of plane angles of contact*, Quart. Appl. Math., 7, pp. 117–120.

(continued from inside front cover)

Jerrold E. Marsden, *Lectures on Geometric Methods in Mathematical Physics*
Bradley Efron, *The Jackknife, the Bootstrap, and Other Resampling Plans*
M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*
D. H. Sattinger, *Branching in the Presence of Symmetry*
R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*
Miklós Csörgo, *Quantile Processes with Statistical Applications*
J. D. Buckmaster and G. S. S. Ludford, *Lectures on Mathematical Combustion*
R. E. Tarjan, *Data Structures and Network Algorithms*
Paul Waltman, *Competition Models in Population Biology*
S. R. S. Varadhan, *Large Deviations and Applications*
Kiyosi Itô, *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*
Alan C. Newell, *Solitons in Mathematics and Physics*
Pranab Kumar Sen, *Theory and Applications of Sequential Nonparametrics*
László Lovász, *An Algorithmic Theory of Numbers, Graphs and Convexity*
E. W. Cheney, *Multivariate Approximation Theory: Selected Topics*
Joel Spencer, *Ten Lectures on the Probabilistic Method*
Paul C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces*
Charles K. Chui, *Multivariate Splines*
Herbert S. Wilf, *Combinatorial Algorithms: An Update*
Henry C. Tuckwell, *Stochastic Processes in the Neurosciences*
Frank H. Clarke, *Methods of Dynamic and Nonsmooth Optimization*
Robert B. Gardner, *The Method of Equivalence and Its Applications*
Grace Wahba, *Spline Models for Observational Data*
Richard S. Varga, *Scientific Computation on Mathematical Problems and Conjectures*
Ingrid Daubechies, *Ten Lectures on Wavelets*
Stephen F. McCormick, *Multilevel Projection Methods for Partial Differential Equations*
Harald Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*
Joel Spencer, *Ten Lectures on the Probabilistic Method, Second Edition*
Charles A. Micchelli, *Mathematical Aspects of Geometric Modeling*
Roger Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis, Second Edition*
Glenn Shafer, *Probabilistic Expert Systems*
Peter J. Huber, *Robust Statistical Procedures, Second Edition*
J. Michael Steele, *Probability Theory and Combinatorial Optimization*